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Terminological Logics with Modal Operators

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Abstract

Terminological knowledge representation formalisms can be used to represent objective, time-independent facts about an application domain. Notions like belief, intentions, time—which are essential for the representation of multi-agent environments—can only be expressed in a very limited way. For such notions, modal logics with possible worlds semantics provides a formally well-founded and well-investigated basis.

This paper presents a framework for integrating modal operators into terminological knowledge representation languages. These operators can be used both inside of concept expressions and in front of terminological and assertional axioms. The main restrictions are that all modal operators are interpreted in the basic logic K, and that we consider increasing domains instead of constant domains. We introduce syntax and semantics of the extended language, and show that satisfiability of finite sets of formulas is decidable.

Contents

1 Introduction

Terminological knowledge representation languages in the style of KL-ONE [7] have been developed as a structured formalism to describe the relevant concepts of a problem domain and the interactions between these concepts. Starting with concept names (unary predicates) and roles (binary predicates), one constructs more complex concepts with the help of operators provided by a given concept language. For example, if we have the concept names *Rich*, *Woman* and *Man*, and the atomic roles *loves*, we can describe the concept "men that love only rich women" by the expression $Man \sqcap \forall loves.(\neg Woman \sqcup Rich)$. Concept definitions (terminological axioms) can then be used to introduce names for complex concepts. For example, "fortune hunter" could be an appropriate name for the concept from above:

$$Fortune\text{-}hunter = Man \sqcap \forall loves.(\neg Woman \sqcup Rich).$$

In addition, so-called assertional axioms can be used to associate objects (or individuals) with concepts and to describe role relationships between objects. For example, one can say that Peter is a fortune hunter, who loves Mary, using the assertions

$$Peter : Fortune\text{-}hunter \quad \text{and} \quad Peter\ loves\ Mary.$$

Various terminological systems have been designed and implemented that are based on the ideas underlying KL-ONE, for example, BACK [21], CLASSIC [6], KANDOR [20], KL-TWO [27] K-Rep [18], KRYPTON [5] KRIS [3], LOOM [17], MESON [9], NIKL [12], SB-ONE [14].

Representing knowledge of an application domain with such a system amounts to introducing the terminology of this domain via concept definitions, and then describing (an abstraction of) the relevant part of the "world" by listing the facts that hold in this part of the world. In a traditional terminological system, such a description is rigid in the sense that it does not allow for the representation of notions like time, or beliefs of different agents. Thus, in a pure terminological formalism we can express the fact that "Mary loves John," but we are unable to formulate facts like "John believes that Mary loves him" or "until yesterday, Mary loved John". In systems modeling aspects of intelligent agents, however, intentions, beliefs, and time-dependent facts play an important role.

Modal logics with possible worlds semantics is a formally well-founded and well-investigated framework for the representation of such notions. The present paper is concerned with integrating modal operators (for time, belief, etc.) into a terminological formalism. For example, if we extend the terminological language by two modal operators $[belief\text{-}John]$ and $\langle future \rangle$, to be read as "John believes that" and "at some time in the future it will hold that," respectively, we can use a formula like

$$[belief\text{-}John]\,(\psi \wedge \langle future \rangle\, \phi)$$

to represent the fact that John believes that Mary does not love him yet (expressed by the assertion ψ : *Mary loves Peter*), but he thinks that this will change eventually (expressed by the assertion ϕ : *Mary loves John*). This small example shows that there could be a high degree of cross-fertilization between terminological knowledge representation and modal logics. For this to come true one must find an appropriate semantics for the combined language. In addition, if such a language should be used in a system, one must design algorithms for the important inference problems (such as consistency of knowledge bases) for the language.

Several approaches have been proposed to combine terminological formalisms with notions like time or beliefs. A very simple possibility to represent beliefs of agents is realized in the partition hierarchy SB-PART [13], which is an extension of the SB-ONE system. In this approach, each agent may have its own set of terminological axioms (*TBox*), and these TBoxes can be ordered hierarchically. However, this extension lacks a formal semantics and it does not allow for representing properties of belief, such as introspection, or interactions between beliefs of different agents. A more formal approach is used in M-KRYPTON [22], where a sub-language of the KRYPTON representation language is extended by modal operators B_i, which can be used to represent the beliefs of agent i. Properties of beliefs are taken into consideration by using the well-known modal logic $\mathsf{KD45}$. Due to the undecidable base language, however, [22] just introduces a formal semantics, without giving any inference algorithms for the extended language. In [23], it has been shown that terminological systems already have a strong connection to modal logic. In fact, the concept language $\mathcal{ALC}$ is nothing but a syntactic variant of the propositional multi-modal logic $\mathsf{K}_{(m)}$. Building upon this observation, [24] augments $\mathcal{ALC}$ by tense operators.

The two approaches that come next to the one we shall introduce below are described in [15, 16] and in [19]. Both extend $\mathcal{ALC}$ by modal operators, but with different emphasis. In [15, 16], modal operators are allowed in front of terminological and assertional axioms only, and not inside of concept expressions. In [19], multi-modal operators can be used at all levels of the concept expressions and, additionally, they can be used to modify roles and other modal operators. However, assertional axioms have not been considered at all, and terminological axioms (concept definitions) are only provided with a semantics, but they are not treated by the inference algorithm described in [19].

2 Classification

When extending a terminological KR language by modalities for belief, time, etc. one has various degrees of freedom. Before describing the specific choices made in this article, we shall informally explain the different alternatives. This will also clarify the differences between the formalism presented below and the extensions of terminological languages described in [19] and [15, 16].

For simplicity, assume that we are interested in time and belief operators only. Thus, in addition to the objects (like John and Mary) we have time points and belief worlds. This means that the domain of an interpretation is the Cartesian product $\vec{D} = D_{object} \times D_{time} \times D_{belief}$ of the set of objects, the set of time points, and the set of belief worlds. Concepts are no longer just sets of objects; their interpretation also depends on the actual belief world and time point. Thus, they can be seen as subsets of $\vec{D}$, and not just as subsets of D_{object}. Roles (like *loves* or *owns*) operate on objects, whereas modalities for time (like *future* or *tomorrow*) operate on time points, and modalities for belief (like *belief-John*) operate on belief worlds. As for concepts, however, the interpretation of roles and modalities depends on all dimensions. Thus, *loves* is interpreted as a function from $\vec{D}$ into $2^{D_{object}}$, which relates any individual in D_{object} (say John) with a set of individuals (the individuals John loves), but this set depends on the actual time point and belief world. Modalities like *future* are treated analogously. Modal operators can now be used both inside of concept expressions and in front of concept definitions and assertions. For example, we can describe the set of individuals that love a woman that—according to John's belief—is pretty by the concept expression

$$\exists\, loves.(Woman \sqcap [belief\text{-}John] Pretty),$$

and we can express that—according to John's belief—a happy husband is one married to a woman whom he (John) believes to be pretty by

$$[belief\text{-}John]\,(Happy\text{-}husband = \exists\, married\text{-}to.(Woman \sqcap [belief\text{-}John] Pretty))\,.$$

The assertion $[belief\text{-}John]\langle future\rangle\ (Peter\ married\text{-}to\ Mary)$ says that John believes that, at some point in the future, Peter will be married to Mary.

With the usual interpretation of the Boolean operators, of value and exists restrictions on roles, and of box and diamond operators for the modalities, this yields a multi-dimensional version of the multi-modal logic K_m. As described until now, this logic is a sub-language of the one introduced in [19]. The restriction lies in the fact that, unlike in [19], we do not consider roles and modalities that have a complex structure (such as $[wants]own$, where the modality *wants* is used to modify the role *own*).

There are several reasons why this approach is not yet satisfactory. First, the object and the other dimensions are treated analogously. This means, for example, that the interpretation of the modality *future* depends not only on the actual time point, but also on the current object and the belief world. Whereas the dependence from the belief world may seem to be quite reasonable, it is rather counterintuitive that the future time points reached from time t_0 are different, depending on whether we are interested in the individual Sue or Mary. For example, assume that for all time points, Sue belongs to the interpretation of *Pretty* iff Mary belongs to the interpretation of *Pretty*. Nevertheless, it could be the case that Mary belongs to $\langle future\rangle Pretty$, whereas

Sue does not. In fact, in the future time point at which Mary is pretty, Sue is pretty as well. However, this time point may only be a future time point with regard to Mary, but not with regard to Sue. Thus, it seems to be more appropriate to treat the object dimension in a special way: whereas the interpretation of roles should depend on the actual time point etc., the interpretation of modalities should not depend on the object under consideration.

The need for a special treatment of the object dimension can also be motivated by considering the semantics of concept definitions (and assertions). In [19], concept definitions are required to hold for all objects, time points, and belief worlds. This is a straightforward generalization of the treatment of definitions in terminological languages, where a definition $C = D$ must hold for *all* objects, i.e., in a model of $C = D$ all objects o must satisfy that o belongs to the interpretation of C iff it belongs to the interpretation of D. For the other dimensions, however, this differs from the usual definition of models in modal logics, where a formula is only required to hold in one world. (Only the characteristic axioms of the particular modal system are required to hold in all worlds.)

Another problem is that not only the roles, but also all the other modalities are just interpreted in the basic logic K, i.e., they are not required to satisfy specific axioms for belief or time.

In the present paper, we shall not take into account this last aspect, but we shall treat the object dimension in a special way, thus eliminating the problems mentioned above. In [15, 16] both aspects are considered. However, modal operators are not allowed to occur inside of concept expressions, which considerably simplifies the algorithmic treatment of the formalism. The difference to [19] is, on the one hand, the special treatment of the object dimension. In addition, [19] does not consider assertions, and even though concept definitions are introduced, they are not handled by the satisfiability algorithm. On the other hand, [19] allows for very complex roles and modalities, which are not considered here.

3 Syntax and Semantics of $\mathcal{ALC}_{\mathcal{M}}$

First, we present the syntax of our multi-dimensional modal extension of the concept language $\mathcal{ALC}$. As for $\mathcal{ALC}$, we assume a set of concept names, a set of role names, and a set of object names to be given. Beside the object dimension (which will be treated differently from the other dimensions), we assume that there are $\nu \geq 1$ additional dimensions (such as time points, epistemic alternatives, or intensional states). In each dimension, there can be several modalities, which can be used in box and diamond operators. For example, in the dimension *time points* we could have *future* and *tomorrow*, and in the dimension *belief worlds* we could have *belief-John* and *belief-Mary*. If

o is a modality of dimension i we write $dim(o) = i$. In this case, $[o]$ and $\langle o \rangle$ are *modal operators* of dimension i.

Definition 3.1 Concept descriptions *(or, for short,* concepts*) of* $\mathcal{ALC}_\mathcal{M}$ *are inductively defined as follows:*

1. *Each concept name is a concept, and* $\top$ *and* $\bot$ *are concepts.*
2. *If* C *and* D *are concepts,* R *is a role name, and* o *is a modality then*
   - *(a)* $C \sqcap D$ (concept conjunction), $C \sqcup D$ (concept disjunction), *and* $\neg C$ (concept negation) *are concepts,*
   - *(b)* $\forall R.C$ (value restriction) *and* $\exists R.C$ (exists restriction) *are concepts,*
   - *(c)* $[o]\, C$ (box operator) *and* $\langle o \rangle\, C$ (diamond operator) *are concepts.*

Terminological axioms *of* $\mathcal{ALC}_\mathcal{M}$ *are of the form* $m\,(C = D)$ *where* C *and* D *are concepts of* $\mathcal{ALC}_\mathcal{M}$ *and* m *is a (possibly empty) sequence of modal operators.* Assertional axioms *of* $\mathcal{ALC}_\mathcal{M}$ *are of the form* $m\,(xRy)$ *or* $m\,(x : C)$ *where* x *and* y *are object names,* R *is a role name,* C *is a concept, and* m *is a (possibly empty) sequence of modal operators. An* $\mathcal{ALC}_\mathcal{M}$-formula *is either a terminological or an assertional axiom.*

Traditional terminological systems impose severe restrictions on the admissible sets of terminological axioms: *(1)* The concepts on the left-hand sides of axioms must be concept names, *(2)* concept names occur at most once as left-hand side of an axiom (unique definitions), and *(3)* there are no cyclic definitions. The effect of these restrictions is that terminological axioms are just macro definitions (introducing names for large descriptions), which can simply be expanded before starting the reasoning process. Unrestricted terminological axioms are a lot harder to handle algorithmically [25, 2, 8], but they are very useful for expressing constraints on concepts that are required to hold in the application domain. In the presence of modal operators, the requirement of having unique definitions is not appropriate anyway. For example, Peter may have a definition of *Happy-husband* that is quite different from John's definition. Thus, it is desirable to have different definitions $m_1\,(A = C)$ and $m_2\,(A = D)$ of the same concept name A for different modal sequences m_1 and m_2. Even though m_1 and m_2 are different, there can be interactions between these definitions. For example, m_1 could be of the form $\langle o \rangle$ and m_2 of the form $[o]$. Thus, it is not a priori clear how the requirement of "unique definitions" can be adapted to case of terminological axioms with modal prefix. To avoid these problems, we consider the more general case where arbitrary axioms are allowed.

Let us now turn to the semantics of $\mathcal{ALC}_\mathcal{M}$. The modal operators will be interpreted by a Kripke-style possible worlds semantics [11]. Thus, for each dimension i we need a set of possible worlds D_i. Modalities of dimension i correspond to accessibility relations on D_i, which may, however, depend on the other dimensions as well. Concepts and roles are interpreted in an object domain, but this interpretation also depends on the modal dimensions. The next definition formalizes these ideas.

Definition 3.2 *A* Kripke structure $K = (\mathcal{W}, \Gamma, K_I)$ *consists of a set* $\mathcal{W}$ *of possible worlds, a set of accessibility relations* Γ*, and a* K*-interpretation* K_I *over* $\mathcal{W}$*:*

- $\mathcal{W}$ *is the Cartesian product of non-empty domains* $D_1, \ldots, D_\nu$*, one for each dimension. It will be called the set of possible worlds.*
- Γ *contains for each modality* o *of dimension* i *an accessibility relation* γ_o*, which is a function* $\gamma_o : \mathcal{W} \to 2^{D_i}$*. Instead of* $d'_i \in \gamma_o(d_1, \ldots, d_i, \ldots, d_\nu)$ *we will often write* $((d_1, \ldots, d_i, \ldots, d_\nu), (d_1, \ldots, d'_i, \ldots, d_\nu)) \in \gamma_o$*.*
- *The* K*-interpretation* K_I *consists of a domain* Δ^{K_I} *and an interpretation function* $\cdot^{K_I}$*. The domain is the union of non-empty domains* $\Delta^{K_I}(w)$ *for all worlds* $w \in \mathcal{W}$*. The interpretation function associates*
  - *with each object name* x *an element* $x^{K_I} \in \Delta^{K_I}$*,*
  - *with each concept name* A *and world* $w \in \mathcal{W}$ *a set* $(A, w)^{K_I} \subseteq \Delta^{K_I}(w)$*,*
  - *with the top concept and the bottom concept the sets* $(\top, w)^{K_I} = \Delta^{K_I}(w)$ *and* $(\bot, w)^{K_I} = \emptyset$ *(for each world* w*),*
  - *with each role name* R *and world* $w \in \mathcal{W}$ *a binary relation* $(R, w)^{K_I} \subseteq \Delta^{K_I}(w) \times \Delta^{K_I}(w)$*.*

Note that the interpretation of object names does not depend on the particular world (i.e., we are using so-called "rigid designators"), whereas the interpretation of concept and role names depends on the world. For a given world w, the interpretation of A (resp. R) in w is a subset of (resp. binary relation on) the domain $\Delta^{K_I}(w)$ associated with w.

The interpretation of concept names and roles is expanded to the concept forming operators as follows: If C and D are concepts, R is a role, and w is a world, then

$$
\begin{aligned}
(C \sqcap D, w)^{K_I} &= (C, w)^{K_I} \cap (D, w)^{K_I}, \\
(C \sqcup D, w)^{K_I} &= (C, w)^{K_I} \cup (D, w)^{K_I}, \\
(\neg C, w)^{K_I} &= \Delta^{K_I}(w) \setminus (C, w)^{K_I}, \\
(\forall R.C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta' \in (C, w)^{K_I} \text{ for each } \delta' \text{ with } (\delta, \delta') \in (R, w)^{K_I}\}, \\
(\exists R.C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta' \in (C, w)^{K_I} \text{ for some } \delta' \text{ with } (\delta, \delta') \in (R, w)^{K_I}\}, \\
([o]\, C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta \in (C, w')^{K_I} \text{ for each world } w' \text{ with } (w, w') \in \gamma_o\}, \\
(\langle o \rangle\, C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta \in (C, w')^{K_I} \text{ for some world } w' \text{ with } (w, w') \in \gamma_o\}.
\end{aligned}
$$

Note that, for each concept C and world w, we have $(C, w)^{K_I} \subseteq \Delta^{K_I}(w)$. Two $\mathcal{ALC}_\mathcal{M}$ concepts C and D are called *equivalent* iff for all Kripke structures $K = (\mathcal{W}, \Gamma, K_I)$ and all worlds $w \in \mathcal{W}$ we have $(C, w)^{K_I} = (D, w)^{K_I}$.

Now, we can define under which condition an $\mathcal{ALC}_{\mathcal{M}}$-formula F is *satisfied in a Kripke structure* $K = (\mathcal{W}, \Gamma, K_I)$ *and a world* $w \in \mathcal{W}$, written as $K, w \models F$, by induction on the length of the modal prefix:

$$\begin{array}{lll} K, w \models C = D & \text{iff} & (C, w)^{K_I} = (D, w)^{K_I}, \\ K, w \models x : C & \text{iff} & x^{K_I} \in (C, w)^{K_I}, \\ K, w \models xRy & \text{iff} & (x^{K_I}, y^{K_I}) \in (R, w)^{K_I}, \\ K, w \models [o]\, G & \text{iff} & K, w' \models G \text{ for each world } w' \text{ with } (w, w') \in \gamma_o, \\ K, w \models \langle o \rangle\, G & \text{iff} & K, w' \models G \text{ for some world } w' \text{ with } (w, w') \in \gamma_o. \end{array}$$

Here G is an $\mathcal{ALC}_{\mathcal{M}}$-formula, C, D are concepts, x, y are object names, R is a role name, and o is a modality. A set $\{F_1, \ldots, F_n\}$ of $\mathcal{ALC}_{\mathcal{M}}$-formulas is *satisfiable* iff there exists a Kripke structure $K = (\mathcal{W}, \Gamma, K_I)$ and a world $w_0 \in \mathcal{W}$ such that $K, w_0 \models F_i$ for $i = 1, \ldots, n$. In this case we write $K \models F_1, \ldots, F_n$.

Even though we have introduced a domain $\Delta^{K_I}(w)$ for each world w, we have not yet said anything about the relationship between the domains of different worlds. In the simplest approach, the domains $\Delta^{K_I}(w_1)$ and $\Delta^{K_I}(w_2)$ of each pair w_1, w_2 of worlds are independent of each other. This approach is known as *varying domain assumption*. In most cases, however, it is more reasonable to assume certain relationships between the domains of different worlds.

The most commonly used approach is the so-called *increasing domain assumption*, where $\Delta^{K_I}(w) \subseteq \Delta^{K_I}(w')$ if the world w' is accessible from the world w. Accessible means that there are $n \geq 1$ worlds $w_1, \ldots, w_n$ such that $w = w_1$, $w' = w_n$, and for all $i, 1 \leq i < n$, there exist a modality o such that $(w_i, w_{i+1}) \in \gamma_o$. The advantage of this approach is that domain elements that have been introduced in w can also be referred to in all worlds that are accessible from w, i.e., domain elements do not "vanish" when we move from one world to another. As an example, consider worlds as time points, and the accessibility relation between worlds as the flow of time. With increasing domain assumption, if there is a domain element *Aristotle* at some time point t_1, we can speak about *Aristotle* at any time point later than t_1 (i.e., which is accessible from t_1).

As a special case, the *constant domain assumption* is sometimes used, where the domains $\Delta^{K_I}(w_1)$ and $\Delta^{K_I}(w_2)$ are identical whenever world w_2 is accessible from w_1. Finally, the *decreasing domain assumption* can be used to express that new domain elements cannot arise when moving from one world to another one.

As an example that demonstrates the consequences of changing the requirements on the relationship between domains of worlds, consider the $\mathcal{ALC}_{\mathcal{M}}$-formulas $x : (\langle o \rangle\, C)$ and $\langle o \rangle\, (x : C)$, where x is an object name, o is a modality, and C is a concept. For a Kripke structure $K = (\mathcal{W}, \Gamma, K_I)$ and a world $w \in \mathcal{W}$ we have

(*i*) $K, w \models x : \langle o \rangle\, C$ iff $x^{K_I} \in \Delta^{K_I}(w)$ and there exists a world w' such that $(w, w') \in \gamma_o$ and $x^{K_I} \in (C, w')^{K_I}$, (*ii*) $K, w \models \langle o\rangle(x:C)$ iff there exists a world w' such that $(w, w') \in \gamma_o$, $x^{K_I} \in \Delta^{K_I}(w')$, and $x^{K_I} \in (C, w')^{K_I}$.

Thus, the main difference is that in the first case x^{K_I} is required to be in $\Delta^{K_I}(w)$, whereas this is not necessary in the second case. The reason is that, in the first case, x must belong to the interpretation of a concept in world w. In the second case, x is just required to be in the interpretation of a concept in the successor world.

If we assume just increasing domains, it is possible that $x^{K_I} \in \Delta^{K_I}(w')$, but $x^{K_I} \notin \Delta^{K_I}(w)$. Hence it may be the case that $K, w \models \langle o\rangle\,(x:C)$, but $K, w \not\models x:(\langle o\rangle\, C)$. If we assume constant domains, however, it holds that $\Delta^{K_I}(w) = \Delta^{K_I}(w')$, and thus $K, w \models x:(\langle o\rangle\, C)$ iff $K, w \models \langle o\rangle\,(x:C)$.

With the exception of Section 6, where we discuss the algorithmic problems that are caused by the constant domain assumption, we will restrict our attention to increasing domains in the following. Furthermore, we assume that all terminological axioms are of the form $m\,(C = \top)$, where C is a concept and m is a (possibly empty) sequence of modal operators. It is easy to verify that this can be done without loss of generality.

Lemma 3.3 *Let* $K = (\mathcal{W}, \Gamma, K_I)$ *be a Kripke structure,* w *be a world in* $\mathcal{W}$*,* m *be a (possibly empty) sequence of modal operators, and* C, D *be concepts. Then* $K, w \models m\,(C = D)$ *iff* $K, w \models m\,\big((C \sqcap D) \sqcup (\neg C \sqcap \neg D) = \top\big)$.

Proof: First, assume that m is empty. Then $K, w \models C = D$ iff $(C, w)^{K_I} = (D, w)^{K_I}$. This is equivalent to saying that, for each element $\delta \in \Delta^{K_I}(w)$, it holds that either (*i*) $\delta \in (C, w)^{K_I}$ and $\delta \in (D, w)^{K_I}$ or (*ii*) $\delta \notin (C, w)^{K_I}$ and $\delta \notin (D, w)^{K_I}$. This is the case iff $K, w \models (C \sqcap D) \sqcup (\neg C \sqcap \neg D)$. Building upon this, the argument is straightforward for non-empty modal prefix m. □

4 Testing Satisfiability of $\mathcal{ALC}_{\mathcal{M}}$-formulas

We present an algorithm for testing satisfiability of a finite set $\{F_1, \ldots, F_n\}$ of $\mathcal{ALC}_{\mathcal{M}}$-formulas.[1] To keep notation simple we assume concepts to be in *negation normal form*, i.e., negation signs occur immediately in front of concept names only. Concepts can be transformed into an equivalent negation normal form by the rules

$$\begin{array}{lll}
\neg[o]\,C \rightarrow \langle o\rangle\,\neg C & \neg\neg C \rightarrow C & \neg(C \sqcap D) \rightarrow \neg C \sqcup \neg D \\
\neg\langle o\rangle\,C \rightarrow [o]\,\neg C & \neg\top \rightarrow \bot & \neg(C \sqcup D) \rightarrow \neg C \sqcap \neg D \\
 & \neg\bot \rightarrow \top & \neg(\forall\, R.C) \rightarrow \exists\, R.\neg C \\
 & & \neg(\exists\, R.C) \rightarrow \forall\, R.\neg C
\end{array}$$

[1] It is easy to see that all the other interesting inference problems (like the subsumption or the instance problem) can be reduced to this problem.

where o is a modality, C is a concept, and R is a role. Our calculus for testing satisfiability of $\mathcal{ALC}_\mathcal{M}$-formulas is based on the notions of labeled $\mathcal{ALC}_\mathcal{M}$-formulas and of world constraint systems.

Definition 4.1 *A* labeled $\mathcal{ALC}_\mathcal{M}$-formula *consists of an $\mathcal{ALC}_\mathcal{M}$-formula F together with a label l, written as $F \,||\, l$. The label l is a syntactic representation of a world in which F is required to hold. A* world constraint *is either a labeled $\mathcal{ALC}_\mathcal{M}$-formula or a term $l \bowtie_o l'$, where l, l' are labels and $\bowtie_o$ is a syntactic representation of the accessibility relation of modality o. A* world constraint system *is a finite, non-empty set of world constraints.*

A Kripke structure $K = (\mathcal{W}, \Gamma, K_I)$ satisfies a world constraint system W iff there is a mapping α that maps labels in W to worlds in $\mathcal{W}$ such that (i) $K, \alpha(l) \models F$ for each world constraint $F \,||\, l$ in W, and (ii) $(\alpha(l), \alpha(l')) \in \gamma_o$ for each world constraint $l \bowtie_o l'$ in W. A world constraint system W is *satisfiable* iff there exists a Kripke structure satisfying W.

In order to test satisfiability of a set $\{F_1, \ldots, F_n\}$ of $\mathcal{ALC}_\mathcal{M}$-formulas we translate this set into the world constraint system $W_0 = \{x_0 : \top \,||\, l_0, F_1 \,||\, l_0, \ldots, F_n \,||\, l_0\}$, where x_0 is a new object name not occurring in $\{F_1, \ldots, F_n\}$, and l_0 is an arbitrary label (which is intended to represent the real world). We say the world constraint system W_0 is *induced by* $\{F_1, \ldots, F_n\}$. It is easy to verify that $\{F_1, \ldots, F_n\}$ is satisfiable iff W_0 is satisfiable. The world constraint $x_0 : \top \,||\, l_0$ can obviously be satisfied by any Kripke structure. The proof of completeness given in the next section will show that this constraint is necessary to guarantee that the domains $\Delta^{K_I}(w)$ of the canonical Kripke structure constructed in this proof are non-empty.

The $\mathcal{ALC}_\mathcal{M}$-satisfiability algorithm takes as input a world constraint system W_0 that is induced by a finite set of $\mathcal{ALC}_\mathcal{M}$-formulas. It successively adds new world constraints to W_0 by applying several propagation rules, which will be defined later. A world constraint system that is induced by a finite set of $\mathcal{ALC}_\mathcal{M}$-formulas, or that is obtained by a finite sequence of applications of propagation rule to an induced system, will be called *derived system*.

In the following, we use the letters x, y, z to denote object names, l to denote labels, A, B to denote concept names, C, D to denote concepts, and R to denote role names. If necessary, these letters will have an appropriate subscript.

Before introducing the rules in a formal way, let us first describe the underlying ideas on an intuitive level. The rules that handle the usual $\mathcal{ALC}$ concept forming operators are well-known and rather straightforward (see, e.g., [4]). As an example for the treatment of the Boolean operators, assume that there is a world constraint $x : C \sqcap D \,||\, l$ in a world constraint system W. The $\rightarrow_\sqcap$ rule adds the world constraints $x : C \,||\, l$ and $x : D \,||\, l$ to W (unless they are already present in W).

Example 4.2 To illustrate the rules that handle modalities and world constraints of the form $C = \top \,||\, l$, suppose that the $\mathcal{ALC}_{\mathcal{M}}$-formula $\langle o \rangle\,(B = \top)$ is given, where o is a modality of some dimension. In order to test satisfiability of this $\mathcal{ALC}_{\mathcal{M}}$-formula, we start with the induced world constraint system

$$W_0 = \{x_0 : \top \,||\, l_0, \langle o \rangle\,(B = \top) \,||\, l_0\}.$$

By definition, W_0 is satisfiable iff there is a Kripke structure $K = (\mathcal{W}, \Gamma, K_I)$, a mapping α, and a world $w_0 = \alpha(l_o) \in \mathcal{W}$ such that $x_0^{K_I} \in \Delta^{K_I}(w_0)$ and $K, w_0 \models \langle o \rangle\,(B = \top)$. Since $K, w_0 \models \langle o \rangle\,(B = \top)$ iff $K, w_1 \models B = \top$ for some world w_1 with $(w_0, w_1) \in \gamma_o$, the $\rightarrow_\diamond$ rule adds the world constraints $l_0 \bowtie_o l_1$ and $B = \top \,||\, l_1$ to W_0, where l_1 is a new label. This yields the new world constraint system

$$W_1 = W_0 \cup \{l_0 \bowtie_o l_1, B = \top \,||\, l_1\}.$$

Because of the semantics of $\mathcal{ALC}_{\mathcal{M}}$-formulas we know that $K, w_1 \models B = \top$ iff $\delta \in (B, w_1)^{K_I}$ for all $\delta \in \Delta^{K_I}(w_1)$. By the increasing domain assumption, $x_0^{K_I} \in \Delta^{K_I}(w_0)$ implies $x_0^{K_I} \in \Delta^{K_I}(w_1)$. Summing up, we must guarantee that $x_0^{K_I} \in (B, w_1)^{K_I}$ and therefore must add the world constraint $x_0 : B \,||\, l_1$ to W.

More generally, we say that an object name x is *relevant for label* l (in a world constraint system W) iff there is a label l' occurring in W such that

1. W contains a world constraint of the form $x : C \,||\, l'$, $xRy \,||\, l'$, or $yRx \,||\, l'$.
2. l is *accessible from* l', i.e., l is l' or there are world constraints $l' \bowtie_{o_1} l_1, \ldots, l_{n-1} \bowtie_{o_n} l$ in W for some modalities $o_1, \ldots, o_n$.

Now, if x is relevant for l and there is a world constraint $C = \top \,||\, l$ in W for some concept C, then the $\rightarrow_=$ rule adds $x : C \,||\, l$ to W (unless this world constraint is already contained in W).

In our example, this rule applies to W_1, and it yields the world constraint system

$$W_2 = W_1 \cup \{x_0 : B \,||\, l_1\}.$$

To W_2 no more propagation rules are applicable, and—as we shall show below—we can use this system to construct a Kripke structure that satisfies the $\mathcal{ALC}_{\mathcal{M}}$-formula $\langle o \rangle\,(B = \top)$ from W_2. A world constraint system to which no more propagation rules are applicable will be called *complete*.

Termination of the propagation rule applications can only be guaranteed if applicability of the usual rule for handling exists restrictions is restricted in an appropriate way. This is due to the presence of axioms of the form $C = \top$.

Example 4.3 To illustrate this problem, consider the world constraint system $W = \{x : A \mid\mid l, \exists R.C = \top \mid\mid l\}$. Since x is relevant for l, the $\rightarrow_=$ rule adds $x : \exists R.C \mid\mid l$. Now, the usual propagation rule $\rightarrow_\exists$ that treats exists restrictions would add $xRy \mid\mid l$ and $y : C \mid\mid l$ to W, where y is a new object. However, y is again relevant for l, and thus we must add $y : \exists R.C \mid\mid l$. The $\rightarrow_\exists$ rule would thus be applicable to $y : \exists R.C \mid\mid l$, generating new world constraints $yRz \mid\mid l$ and $z : C \mid\mid l$, etc.

In order to avoid such infinite chains of rule application, we introduce the notion of blocked objects.[2] Intuitively, an object x is blocked w.r.t. label l if we need not introduce a new object in order to be sure that the exists restrictions on x can be satisfied.

Example 4.4 Consider the world constraint system

$$W = \{x : \exists R.C \mid\mid l, x : D \mid\mid l, xRy \mid\mid l, y : \exists R.C \mid\mid l\}.$$

In this case, it is sufficient to apply the $\rightarrow_\exists$ rule just to x. In fact, since all constraints for y are also constraints for x, any contradiction that could be obtained by applying this propagation rule to y can already be obtained by applying it to x.

The idea is thus to say that y is blocked by x with respect to a label l if $\{C \mid x : C \mid\mid l \in W\} \subseteq \{D \mid y : D \mid\mid l \in W\}$. In the above example, y would thus be blocked by x, and the $\rightarrow_\exists$ rule would only be applied to x. In general, this notion of blocking is too strong, though. In fact, consider the system W' that is obtained from W by deleting the constraint $x : D \mid\mid l$. In this system, x would be blocked by y and vice versa. Such cyclic blocking is clearly not appropriate since contradictions that are possibly hidden in C would never be detected.

In order to avoid cyclic blocking, we assume that the (countably infinite) set of all object names is given by an enumeration $y_1, y_2, y_3, \ldots$ We write $x < y$ if x comes before y in this enumeration. This ordering is used as follows. Whenever a new object y is introduced by applying the $\rightarrow_\exists$ rule to a world constraint system W, y is chosen such that all objects in W are smaller than y w.r.t. this ordering. In addition, only smaller objects can block a given object.

Definition 4.5 *An object x is* blocked *by an object y w.r.t. label l in a world constraint system W iff* $\{C \mid x : C \mid\mid l \in W\} \subseteq \{D \mid y : D \mid\mid l \in W\}$ and $y < x$.

Now, the $\rightarrow_\exists$ rule is applicable to a world constraint $x : \exists R.C \mid\mid l$ in a world constraint system W only if x is not blocked by some object y w.r.t. l in W.

[2] This idea was already used in [8, 1], with slightly differing definitions of blocked objects.

$W \rightarrow_{\diamond} \{l \bowtie_o l', \varphi' \,||\, l'\} \cup W$

if $\varphi \,||\, l$ is in W, where φ is $\langle o \rangle\, F$ (resp. $x : \langle o \rangle\, C$), φ' is F (resp. $x : C$), there is no label l'' in W such that the world constraints $l \bowtie_o l''$ and $\varphi' \,||\, l''$ are in W, and l' is a new label.

$W \rightarrow_{\Box} \{\varphi' \,||\, l'\} \cup W$

if $\varphi \,||\, l$ and $l \bowtie_o l'$ are in W, where φ is $[o]\, F$ (resp. $x : [o]\, C$), φ' is F (resp. $x : C$), and $\varphi' \,||\, l'$ is not in W.

$W \rightarrow_{\sqcap} \{x : C_1 \,||\, l, x : C_2 \,||\, l\} \cup W$

if $x : C_1 \sqcap C_2 \,||\, l$ is in W and W does not contain both world constraints $x : C_1 \,||\, l$ and $x : C_2 \,||\, l$.

$W \rightarrow_{\sqcup} \{x : D \,||\, l\} \cup W$

if $x : C_1 \sqcup C_2 \,||\, l$ is in W, neither $x : C_1 \,||\, l$ nor $x : C_2 \,||\, l$ is in W, and D is either C_1 or C_2.

$W \rightarrow_{\exists} \{xRy \,||\, l, y : C \,||\, l\} \cup W$

if $x : \exists\, R.C \,||\, l$ is in W, x is not blocked in W, and y is a new object such that $y > z$ for all objects z occurring in W.

$W \rightarrow_{\forall} \{y : C \,||\, l\} \cup W$

if $x : \forall\, R.C \,||\, l$ and $xRy \,||\, l$ are in W and W does not contain the world constraint $y : C \,||\, l$.

$W \rightarrow_{=} \{x : C \,||\, l\} \cup W$

if x is relevant for l, $C = \top \,||\, l$ is in W, and $x : C \,||\, l$ is not in W.

Figure 1: Propagation rules of the $\mathcal{ALC}_{\mathcal{M}}$-satisfiability algorithm.

A formal description of the propagation rules is given in Figure 1. Given a set $\{F_1, \ldots, F_n\}$ of $\mathcal{ALC}_{\mathcal{M}}$-formulas the *$\mathcal{ALC}_{\mathcal{M}}$-satisfiability algorithm* proceeds as follows. Starting with the world constraint system W_0 that is induced by $\{F_1, \ldots, F_n\}$, propagation rules are applied as long as possible.

The transformation rules are *sound* in the sense that, if W is a satisfiable world constraint system, each applicable propagation rule can be applied in such a way that the obtained derived system is satisfiable (see Section 5 for a proof). For the "don't-know" non-deterministic $\rightarrow_{\sqcup}$ rule there are two alternative successor systems, and soundness means that one of them is satisfiable if the original system is satisfiable.[3] For the other rules (which are deterministic), soundness just means that application of the rule transforms a satisfiable system into a new satisfiable system.

Furthermore, given an arbitrary induced world constraint system W_0, only a finite number of propagation rules can successively be applied, starting with W_0 (see also Section 5 for a proof). This means that, after a finite number of propagation rule applications to W_0 we obtain a complete world constraint system (i.e., a system to which no more rules apply), say W'. If W' is satisfiable we can conclude that W_0 is satisfiable (since W_0 is a subset of W'). Otherwise, if W' is unsatisfiable, we can possibly derive another complete world constraint system from W_0 by another choice for the non-deterministic $\rightarrow_\sqcup$ rule. If all the (finitely many) choices lead to an unsatisfiable complete system then soundness of the rules implies that the original system W_0 was unsatisfiable.

Thus, it remains to be shown how satisfiability of a complete world constraint system can be decided.

Definition 4.6 *A world constraint system W contains an* obvious contradiction *(or* clash *for short) if it contains either a pair of labeled $\mathcal{ALC}_\mathcal{M}$-formulas of the form $x : A \,||\, l$ and $x : \neg A \,||\, l$ or a labeled $\mathcal{ALC}_\mathcal{M}$-formula $x : \bot \,||\, l$ (for some object x, concept name A, modality o, and label l).*

Obviously, a world constraint system containing a clash is unsatisfiable. On the other hand, if a system is clash-free and complete then it is satisfiable (this property, which shows *completeness* of the propagation rules, will be proved in the next section). Summing up, we obtain the following theorem, which will be proved as soon as soundness, completeness, and termination of the propagation rules are established.

Theorem 4.7 *Satisfiability of a finite set of $\mathcal{ALC}_\mathcal{M}$-formulas is decidable if we assume increasing domains.*

5 Proofs of Soundness, Completeness, and Termination

In this section we prove Theorem 4.7 by giving proofs for soundness, termination, and completeness of the propagation rules in the Subsections 5.1, 5.2, and 5.3.

[3]Note that this is the only source of "don't-know" non-determinism. The choice of an applicable rule is "don't-care" non-deterministic, i.e., we need not try different orders of rule application.

5.1 Soundness

The following lemma states that the propagation rules are sound.

Lemma 5.1 *Let W be a satisfiable derived system. Then each applicable propagation rule can be applied to W in such a way that the obtained derived system W' is satisfiable.*

Proof: For all rules other than the $\rightarrow_\sqcup$ rule we must show that application of the rule transforms a satisfiable system W into a satisfiable system W'. For the $\rightarrow_\sqcup$ we must show that one of the two systems W', W'' that can be derived by applying this rule is satisfiable, provided that the original system W was satisfiable.

Let $K = (\mathcal{W}, \Gamma, K_I)$ be a Kripke structure that satisfies W, and let α be a mapping that maps labels in W to worlds in $\mathcal{W}$ such that (i) $K, \alpha(l) \models F$ for each world constraint $F \mid\mid l$ in W and (ii) $(\alpha(l), \alpha(l')) \in \gamma_o$ for each world constraint $l \bowtie_o l'$ in W. Since W was assumed to be satisfiable such a Kripke structure and a mapping α exist.

Case 1: $W \rightarrow_\Diamond W' = W \cup \{l \bowtie_o l', F \mid\mid l'\}$. Here the $\rightarrow_\Diamond$ rule was applied to a constraint $\langle o \rangle\ F \mid\mid l$ for some modality o and $\mathcal{ALC}_\mathcal{M}$-formula F. The label l' is a new label that does not occur in W. Since the pair K, α satisfies W we know that $K, \alpha(l) \models \langle o \rangle\ F$, and hence there is a world w' in $\mathcal{W}$ such that $(\alpha(l), w') \in \gamma_o$ and $K, w' \models F$. Thus, we can define α' such that $\alpha'(l') = w'$ and $\alpha'(l'') = \alpha(l'')$ for all labels l'' different from l'. Obviously, K, α' satisfies W'.

Case 2: $W \rightarrow_\Diamond W' = W \cup \{l \bowtie_o l', x : C \mid\mid l'\}$. Here the $\rightarrow_\Diamond$ rule was applied to a constraint $x : \langle o \rangle\ C \mid\mid l$ (for a modality o and an $\mathcal{ALC}_\mathcal{M}$ concept C). Again, label l' is a new label that does not occur in W. Now $K, \alpha(l) \models x : \langle o \rangle\ C$ implies that $x^{K_I} \in (\langle o \rangle\ C, \alpha(l))^{K_I}$, and thus $x^{K_I} \in (C, w')^{K_I}$ for some world w' with $(\alpha(l), w') \in \gamma_o$. As in Case 1, we define $\alpha'(l') = w'$ and $\alpha'(l'') = \alpha(l'')$ for all other labels. Obviously, K, α' satisfies W'.

Case 3: $W \rightarrow_\Box W' = W \cup \{F \mid\mid l'\}$. Here the $\rightarrow_\Box$ rule was applied to $[o]\ F \mid\mid l$ and $l \bowtie_o l'$ for some modality o and $\mathcal{ALC}_\mathcal{M}$-formula F. Since K, α satisfies W we can conclude that $(\alpha(l), \alpha(l')) \in \gamma_o$ and that $K, w' \models F$ for each world w' such that $(\alpha(l), w') \in \gamma_o$. This implies $K, \alpha(l') \models F$, and thus K, α satisfies W'. The case where the $\rightarrow_\Box$ rule is applied to a constraint $x : [o]\ C \mid\mid l$ can be treated similarly.

Case 4: $W \rightarrow_\sqcup W'$, $W \rightarrow_\sqcap W'$, $W \rightarrow_\forall W'$, or $W \rightarrow_\exists W'$. The proof of soundness is almost identical to the one for the corresponding propagation rules for $\mathcal{ALC}$ (see [26]).

Case 5: $W \rightarrow_= W' = W \cup \{x : C \mid\mid l\}$. Here the $\rightarrow_=$ rule was applied to $C = \top \mid\mid l$ and an object name x that is relevant for l in W. Thus, there is a label l' such that (i) W contains a world constraint of the form $x : C \mid\mid l'$, $xRy \mid\mid l'$, or $yRx \mid\mid l'$, and (ii) l is accessible from l'. Because of (i) we know that $x^{K_I} \in \Delta^{K_I}(\alpha(l'))$, and because of (ii) and the increasing domain assumption we have $\Delta^{K_I}(\alpha(l')) \subseteq \Delta^{K_I}(\alpha(l))$.

Furthermore, by assumption, we know that $K, \alpha(l) \models C = \top$, and hence we can conclude $x^{K_I} \in (C, \alpha(l))^{K_I}$. To sum up, $K, \alpha(l) \models x : C$, and thus K, α satisfies W'. $\square$

5.2 Termination

The next lemma shows that, given a finite derived system W, only a finite number of propagation rules can successively be applied to W. In order to simplify the notation we will use $Con_W(x, l)$ to denote the set $\{C_1, \ldots, C_n\}$ of concepts such that $x : C_i \mid\mid l$ occurs in W. By definition, an object x is blocked by an object y w.r.t. label l in W iff $Con_W(x, l) \subseteq Con_W(y, l)$ and $y < x$.

The *depth* of labels in a derived system W is recursively defined as follows. The depth of label l_0 (which represents the real world) is 0, written as $depth_W(l_0) = 0$. If l is a label with $depth_W(l) = n$ and there is a world constraint $l \bowtie_o l'$ in W for some modality o, then $depth_W(l') = n + 1$. Note that, due to the definition of the $\rightarrow_\diamond$ rule, for each label $l' \neq l_0$ occurring a derived system W there is exactly one label l and one modality o such that $l \bowtie_o l'$ is in W. In addition, application of propagation rules does not change the depth of an already existing label.

The *maximal nesting depth* of modal operators in a labeled $\mathcal{ALC}_\mathcal{M}$-formula $F \mid\mid l$ is denoted by $mnd(F \mid\mid l)$. The maximal nesting depth of a label l in a derived system W, written as $mnd_W(l)$, is defined as $max\{mnd(F \mid\mid l) \mid F \mid\mid l \in W\}$.

Lemma 5.2 *Let W_0 be a system that is induced by a finite set of $\mathcal{ALC}_\mathcal{M}$-formulas $\{F_1, \ldots, F_n\}$. Then any sequence of propagation rule applications starting with W_0 is finite.*

Proof: Assume to the contrary there is an infinite sequence of rule applications $W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow \cdots$. The following facts are an easy consequence of the way the propagation rules and *Con* are defined:

1. Let x be an object name in W_i and let l be a label. Then $Con_{W_i}(x, l) \subseteq Con_{W_{i+1}}(x, l)$.
2. If $x : C \mid\mid l$ is in W_i then C is a concept that is a subexpression of a concept occurring in $F_1, \ldots, F_n$. Consequently, there can be only finitely many different sets $Con_{W_i}(x, l)$ (in the whole sequence).
3. If $F \mid\mid l$ is in W_i for an $\mathcal{ALC}_\mathcal{M}$-formula F with non-empty modal prefix then there is an i and a sequence of modal operators m such that $F_i = m\,F$. Consequently, the number of possible formulas F is finite.

The second and the third fact imply that an infinite sequence of rule applications is possible only if infinitely many objects or infinitely many labels are generated. In this case there are three possibilities:

1. For some label l an infinite chain of world constraints of the form $l \bowtie_{o_1} l_1, l_1 \bowtie_{o_2} l_2, \ldots$ is generated,

2. there is a label l such that infinitely many $\mathcal{ALC}_{\mathcal{M}}$-formulas labeled by l are generated, or

3. for some label l an infinite number of world constraints of the form $l \bowtie_{o_1} l_1, l \bowtie_{o_2} l_2, \ldots$ is generated.

First, we show that the first case is impossible. This is an obvious consequence of the following claim. Let mnd_0 be the maximal nesting depth of modal operators in $F_1, \ldots, F_n$. Then for all i and all labels l occurring in W_i we have

$$(*) \quad depth_{W_i}(l) + mnd_{W_i}(l) \leq mnd_0.$$

The claim can be shown by induction on i. For $i = 0$ the only label occurring in W_0 is l_0, and this label has depth 0. In addition, $mnd_{W_0}(l_0) = mnd_0$.

For the induction step, note that it is easy to see that application of rules other than the $\rightarrow_\Diamond$ or $\rightarrow_\Box$ rule cannot change the maximal nesting depth or depth of a label.

First, we consider the case where the world constraint $l \bowtie_o l'$ has been introduced in the step from W_i to W_{i+1} by an application of the $\rightarrow_\Diamond$ rule to a formula φ labeled by l. Thus, we have $depth_{W_{i+1}}(l') = depth_{W_{i+1}}(l) + 1 = depth_{W_i}(l) + 1$. The labeled $\mathcal{ALC}_{\mathcal{M}}$-formula φ is either of the form $\langle o \rangle\ F \mid\mid l$ or of the form $x : \langle o \rangle\ C \mid\mid l$, where F is an $\mathcal{ALC}_{\mathcal{M}}$-formula, x is an object, and C is a concept. If the $\rightarrow_\Diamond$ rule has been applied to $\langle o \rangle\ F \mid\mid l$, then it has added exactly one $\mathcal{ALC}_{\mathcal{M}}$-formula with label l', namely $F \mid\mid l'$. Analogously, if the $\rightarrow_\Diamond$ rule has been applied to $x : \langle o \rangle\ C \mid\mid l$, this propagation rule application has added $x : C \mid\mid l'$ as the only $\mathcal{ALC}_{\mathcal{M}}$-formula with label l'. Thus, in both cases $mnd_{W_{i+1}}(l')$ is strictly smaller than the maximal nesting depth of modal operators in φ. This and the fact that $depth_{W_{i+1}}(l') = depth_{W_i}(l) + 1$ imply that $(*)$ holds for W_{i+1} and l'. Since nothing has changed for the other labels, we are done in this case.

Second, assume that an additional $\mathcal{ALC}_{\mathcal{M}}$-formula ψ with label l' has been added in the step from W_i to W_{i+1} by an application of the $\rightarrow_\Box$ rule to a formula with label l. Again, ψ has a maximal nesting depth of modal operators that is smaller than $mnd_{W_i}(l) = mnd_{W_{i+1}}(l)$, and $depth_{W_{i+1}}(l') = depth_{W_i}(l) + 1$. This implies that $(*)$ holds for W_{i+1} and l'.

This concludes the proof that there cannot be an infinite sequence $l \bowtie_{o_1} l_1, l_1 \bowtie_{o_2} l_2, \ldots$, i.e., Case 1 is not possible. By induction on the depth of labels we show that Cases 2 and 3 cannot occur.

Base Case: Consider the initial label l_0. It is easy to see that Case 2 can occur only if the $\rightarrow_\exists$ rule is applied infinitely often to a constraint with label l_0. To a fixed object x and label l the $\rightarrow_\exists$ rule cannot be applied infinitely many times. This shows that there must be infinitely many objects $x_1, x_2, x_3, \ldots$ to which the $\rightarrow_\exists$ rule is applied at label l_0. Since, for an object x, there are only finitely many smaller objects, we may without loss of generality assume that $x_1 < x_2 < x_3 < \ldots$

For all i, let $W_{j_i} \rightarrow_\exists W_{j_i+1}$ be the transformation step at which the $\rightarrow_\exists$-rule is applied to x_i. Now consider the sets $Con_{W_{j_i}}(x_i, l_0)$. Since there are only finitely many different such sets, there must be indices $k < h$ such that $Con_{W_{j_k}}(x_k, l_0) = Con_{W_{j_h}}(x_h, l_0)$. However, we know that $Con_{W_{j_k}}(x_k, l_0) \subseteq Con_{W_{j_h}}(x_k, l_0)$, and that $x_k < x_h$. Thus, x_h should be blocked in W_{j_h}, which is a contradiction to our assumption that the $\rightarrow_\exists$ rule is applied to x_h in W_{j_h}. This completes the proof that Case 2 cannot hold for label l_0.

An easy consequence of this is that there are only finitely many applications of propagation rules to formulas with label l_0. In particular, the $\rightarrow_\diamond$ rule is applied only a finite number of times to formulas with label l_0, which completes the proof that Case 3 cannot hold for label l_0.

Induction step: The only difference to the base case is that a label l may "inherit" objects from other labels l', if l is accessible from l'. Thus, in addition to showing that the $\rightarrow_\exists$ rule is applied only infinitely often for label l, one must prove that there are only finitely many such inherited objects. However, if l is accessible from l' then l' is of depth smaller than l, and thus we know by induction that there are only finitely many objects for l'. The remainder of the proof is identical to the one for the base case. □

5.3 Completeness

Let $W_0 = \{x_0 : \top \,||\, l_0, F_1 \,||\, l_0, \ldots, F_n \,||\, l_0\}$ be the world constraint system that is induced by the finite set $\{F_1, \ldots, F_n\}$ of $\mathcal{ALC}_\mathcal{M}$-formulas. Assume that W is a complete and clash-free world constraint system that is derived from W_0 by applying propagation rules. We must show that W is satisfiable. Since W_0 is a subset of W, this implies that W_0 is satisfiable, and thus also $\{F_1, \ldots, F_n\}$

In order to show satisfiability of W, we introduce the notion of the *canonical Kripke structure* $K = (\mathcal{W}, \Gamma, K_I)$ of W, and of the corresponding *canonical mapping* α from labels to worlds of this structure.

- For all dimensions i the domain D_i consists of all labels occurring in W, i.e., the set $\mathcal{W}$ of worlds is given by the ν-fold Cartesian product $D \times \cdots \times D$ where $D := \{l \mid l \text{ is a label in } W\}$.
- The mapping α from labels to worlds is defined by induction on the depth of labels:

– The initial label l_0 is the only label of depth 0. For this label, we define $\alpha(l_0) := (l_0, \dots, l_0)$. Obviously, the tuple $\alpha(l_0)$ contains exactly one label of maximal depth (in this case depth 0), and this maximal label is l_0 itself.

– Now assume that l' is a label of depth $k+1$. There is exactly one label l' of depth k and a modality o such that $l \bowtie_o l'$ is in W. Let i be the dimension of o. By induction, we can assume that $\alpha(l)$ is already defined, and that the (unique) label of maximal depth occurring in this tuple is l. The tuple $\alpha(l')$ is obtained from $\alpha(l)$ by replacing the i-th component by l'. Since all the components of $\alpha(l)$ are of depth less or equal k, the unique component of maximal depth in $\alpha(l')$ is l'.

Obviously, the mapping α was defined such that there is a 1–1-correspondence between labels and worlds. Note, however, that not all world tuple are in the image of α. In principle, only those tuples that are in the image are of interest.

- In order to define the accessibility relation γ_o for a given modality o of dimension i we distinguish two cases:

  – If $w \in \mathcal{W}$ is not in the image of α then we set $\gamma_o(w) := \emptyset$.

  – Now, assume that $w = \alpha(l)$ for a label l occurring in W. We define $\gamma_o(w) := \{l' \mid l \bowtie_o l' \in W\}$.

  An easy consequence of this definition and the definition of α is that we have $(w, w') \in \gamma_o$ iff there exist labels l, l' in W such that $w = \alpha(l)$, $w' = \alpha(l')$, and $l \bowtie_o l' \in W$.

- The set Δ^{K_I} consists of all object names occurring in W. For defining the domains of the different worlds, we distinguish two cases:

  – If $w = \alpha(l)$ for a label l then we define $\Delta^{K_I}(w) := \{x \mid x \text{ is relevant for } l\}$. Since every label in W is accessible from l_0, we know that $\Delta^{K_I}(w)$ contains at least the object name x_0.

  – If w is not in the image of α then we set $\Delta^{K_I}(w) := \{x_0\}$.

  It is easy to see that the increasing domain assumption is satisfied this way.

- For each object name x in W we define $x^{K_I} := x$.

- For each concept name A and world $w = \alpha(l)$ we define $(A, w)^{K_I} := \{x \mid x : A \mid\mid l \in W\}$. If w is not in the image of α then $(A, w)^{K_I} := \emptyset$.

- For each role R and world w we define $(R, w)^{K_I} := \emptyset$, if w is not in the image of α. Now, assume that $w = \alpha(l)$. We define $(R, w)^{K_I}$ inductively along the total well-founded ordering $<$ on the object names. If $\widehat{x}$ is the least object w.r.t $<$ we define $(\widehat{x}, y) \in (R, w)^{K_I}$ iff $\widehat{x} R y \mid\mid l \in W$. Now let x be an object in W that is different from $\widehat{x}$.

– If x is not blocked w.r.t. l in W, then $(x, y) \in (R, w)^{K_I}$ iff $xRy \mid\mid l \in W$.
– Otherwise, if x is blocked w.r.t. l in W, let z be the least object (w.r.t. $<$) such that x is blocked by z w.r.t. l. Then $z < x$ and we can thus assume that the set $\{y \mid (z, y) \in (R, w)^{K_I}\}$ is already defined. We define $(x, y) \in (R, w)^{K_I}$ iff (i) $(z, y) \in (R, w)^{K_I}$ or (ii) $xRy \mid\mid l \in W$.

We will show in the following that, given a complete and clash-free derived system W, the canonical Kripke structure of W is a model of W.

Lemma 5.3 *Let W be a complete and clash-free derived system. Then W is satisfiable.*

Proof: Let $K = (\mathcal{W}, \Gamma, K_I)$ be the canonical Kripke structure of W, and let α be the corresponding canonical mapping of labels to worlds. We must show that

1. $(\alpha(l), \alpha(l')) \in \gamma_o$ for each world constraint $l \bowtie_o l'$ in W, and
2. $K, \alpha(l) \models F$ for each labeled $\mathcal{ALC}_{\mathcal{M}}$-formula $F \mid\mid l$ in W.

Because of the definitions of α and of the accessibility relations, the first property is obviously satisfied.

Thus, let us show the second property. If F is of the form xRy there is also nothing to show. In order to treat the other cases, we will first show

$$(*) \quad K, \alpha(l) \models x : C \ \text{ if } x : C \mid\mid l \text{ is in } W$$

by induction on the structure of the concept C.

Case 1 (Base case): If $x : A \mid\mid l$ is in W (where A is a concept name), then $K, \alpha(l) \models x : A$ follows immediately from the construction of K. If $x : \neg A \mid\mid l$ is in W, then we know that A is a concept name because we assumed $\mathcal{ALC}_{\mathcal{M}}$-formulas to be in negation normal form. Since W is clash-free, $x : A \mid\mid l$ is not in W in this case. Thus, $K, \alpha(l) \not\models x : A$ by construction, and therefore $K, \alpha(l) \models x : \neg A$.

Case 2: Assume that $x : C_1 \sqcap C_2 \mid\mid l$ is in W, where C_1 and C_2 are concepts. Since W is complete we know that both labeled $\mathcal{ALC}_{\mathcal{M}}$-formulas $x : C_1 \mid\mid l$ and $x : C_2 \mid\mid l$ are in W. By induction hypothesis it follows that $K, \alpha(l) \models x : C_1$ and $K, \alpha(l) \models x : C_2$, which yield $K, \alpha(l) \models x : C_1 \sqcap C_2$. The argument for $x : C_1 \sqcup C_2$ is analogous.

Case 3: $x : \exists R.C \mid\mid l$ is in W for some concept C. If, for some object y, the world constraints $xRy \mid\mid l$ and $y : C \mid\mid l$ are both in W we can conclude $K, \alpha(l) \models x : \exists R.C$ by definition of R^{K_I} and the induction hypothesis. Now suppose there is no such object y. Then $x : \exists R.C \mid\mid l$ is blocked in W w.r.t. l since W is complete. Let z be the least (w.r.t. $<$) object in W that blocks x. First, note that z is not blocked: otherwise, if

z were blocked by, say z', w.r.t. l then z' would also block x w.r.t. l and $z' < z$. This contradicts the assumption that z is the least object that blocks x.

Since x is blocked by z w.r.t. l, and z is not blocked w.r.t. l, we know that there are world constraints $zR\tilde{z} \mid\mid l$ and $\tilde{z}:C \mid\mid l$ in W. By induction hypothesis, this implies $K, \alpha(l) \models \tilde{z}:C$. In addition, we have $(x, \tilde{z}) \in (R, \alpha(l))^{K_I}$ because of the definition of R^{K_I} in the canonical Kripke structure K. From these facts $K, \alpha(l) \models x:\exists R.C$ can be concluded.

Case 4: $x:\forall R.C \mid\mid l$ is in W. In order to show that $K, \alpha(l) \models x:\forall R.C$ we must show that $y \in (C, \alpha(l))^{K_I}$ for each object y such that $(x, y) \in (R, \alpha(l))^{K_I}$. There are two possibilities for (x, y) to be in $(R, \alpha(l))^{K_I}$, namely (i) there is a world constraint $xRy \mid\mid l$ in W and (ii) x is blocked by some object z w.r.t. l in W—where we assume z to be the least element (w.r.t. $<$) that blocks x in l—and $(z, y) \in (R, l)^{K_I}$. In case (i), the $\rightarrow_\forall$ rule has been applied to $x:\forall R.C \mid\mid l$ and $xRy \mid\mid l$ such that $y:C \mid\mid l$ is in W. In case (ii), we know that $Con_W(x, l) \subseteq Con_W(z, l)$, and hence $z:\forall R.C \mid\mid l$ is in W. Furthermore, since z is not blocked w.r.t. l, we know that $zRy \mid\mid l$ is in W if $(z, y) \in (R, \alpha(l))^{K_I}$. This means, however, that $y:C \mid\mid l$ is in W because otherwise the $\rightarrow_\forall$ rule would be applicable. Now, in both cases $y \in (C, \alpha(l))^{K_I}$ follows from the induction hypothesis.

Case 5: $x:\langle o \rangle C \mid\mid l$ is in W for some modality o. Since W is complete there is a label l' such that the world constraints $l \bowtie_o l'$ and $x:C \mid\mid l'$ are both in W. Consequently, we have $(\alpha(l), \alpha(l')) \in \gamma_o$, and the induction hypothesis yields $K, \alpha(l') \models x:C$. This implies $K, \alpha(l) \models x:\langle o \rangle C$. The proof for $x:[o]\, C \mid\mid l$ is similar.

This completes the proof of $(*)$ by induction on the structure of the concept C in labeled concept instances of the form $x:C \mid\mid l$. Thus, we know that K satisfies each labeled concept instance in W. It remains to be shown that K satisfies each labeled world constraint of the form $\langle o \rangle\, F \mid\mid l$, $[o]\, F \mid\mid l$, or $C = \top \mid\mid l$ in W, where F is an $\mathcal{ALC}_\mathcal{M}$-formula, o is a modality, C is a concept, and l is a label.

First, assume that $C = \top \mid\mid l$ occurs in W. We must show that $x \in (C, \alpha(l))^{K_I}$ for each $x \in \Delta^{K_I}(\alpha(l))$. Since $\Delta^{K_I}(\alpha(l))$ consists of exactly those objects that are relevant for l, and W is complete, the $\rightarrow_=$ rule has been applied and $x:C \mid\mid l$ occurs in W. We have already shown that in this case $K, \alpha(l) \models x:C$ holds.

Second, let $\langle o \rangle\, F \mid\mid l$ be in W. In this case, for some label l' the world constraints $l \bowtie_o l'$ and $F \mid\mid l'$ are in W since W is complete. If F does not contain a leading modality, we have already shown that $K, \alpha(l') \models F$. Otherwise, $K, \alpha(l) \models \langle o \rangle\, F$ can easily been shown by induction on the number of modalities in $\langle o \rangle\, F$. The argument for $[o]\, F \mid\mid w$ is accordingly. □

This completes the proof of Theorem 4.7, i.e., we have shown that satisfiability of a finite set of $\mathcal{ALC}_\mathcal{M}$-formulas is decidable (w.r.t. the increasing domain assumption). A short look at the algorithm reveals that the number ν of different dimensions, and the fact that different modalities may operate on different dimensions was never explicitly used in this algorithm. Thus, if we are only interested in satisfiability, there is no difference between the ν-dimensional formalism (where modalities have different dimensions and the set of worlds is a ν-fold Cartesian product) and the corresponding 1-dimensional language (where we assume just one dimension in which all modalities operate).

Corollary 5.4 *Assume that $\{F_1, \ldots, F_n\}$ is a set of formulas for a ν-dimensional $\mathcal{ALC}_\mathcal{M}$-language. Then $\{F_1, \ldots, F_n\}$ is satisfiable (in the ν-dimensional case) iff it is satisfiable in the corresponding 1-dimensional language.*

6 The Constant Domain Assumption

Up to now we have investigated increasing domains only. In this section we will consider the consequences of assuming that the domains of all worlds are identical. Since this constant domain assumption is a special case of assuming increasing domains, an appropriate extension of the presented $\mathcal{ALC}_\mathcal{M}$-satisfiability algorithm might seem to be rather easy. The goal of this section is to point out why developing such an extended algorithm requires more than a straightforward modification of the existing approach. In fact, until now we did not succeed in finding an appropriate modification.

In a first attempt one could try to use the presented $\mathcal{ALC}_\mathcal{M}$-satisfiability algorithm for the case of constant domains as well. However, not surprisingly, this does not always yield the correct answers. For example, consider the $\mathcal{ALC}_\mathcal{M}$-formulas

$$([o]\,\neg A) = \top \qquad \text{and} \qquad \langle o \rangle\,(x : A)$$

where o is a modality, x an object, and A a concept name. It is easy to see that an application of the $\mathcal{ALC}_\mathcal{M}$-satisfiability algorithm to the induced system $\{x_0 : \top || l_0, ([o]\neg A) = \top\,||\,l_0, \langle o \rangle\,(x : A)\,||\,l_0\}$ yields a complete and clash-free derived system. The reason is that the object name x is not relevant for l_0. This shows that the above $\mathcal{ALC}_\mathcal{M}$-formulas are satisfiable if we assume increasing domains.

On the other hand, they are not satisfiable if we assume constant domains. Suppose, to the contrary, that $K = (\mathcal{W}, \Gamma, K_I)$ is a Kripke structure such that $K, w \models ([o]\,\neg A) = \top$ and $K, w \models \langle o \rangle\,(x : A)$ for some world w in $\mathcal{W}$. Because of $K, w \models \langle o \rangle\,(x : A)$ there exists a world w' with $(w, w') \in \gamma_o$ and $K, w' \models x : A$, i.e. $x^{K_I} \in (A, w')^{K_I}$. On the other hand, we have $x^{K_I} \in (\neg A, w')^{K_I}$ since $x^{K_I} \in \Delta^{K_I}(w)$ (constant domain assumption) and $K, w \models ([o]\,\neg A) = \top$.

In the $\mathcal{ALC}_\mathcal{M}$-satisfiability algorithm we took the increasing domain assumption into consideration by an appropriate definition of the notion of "relevant objects," which was then used in the $\rightarrow_=$ rule: given a labeled $\mathcal{ALC}_\mathcal{M}$-formula $C = \top\,||\,l$ in a derived system W, the $\rightarrow_=$ rule adds the labeled $\mathcal{ALC}_\mathcal{M}$-formula $x\!:\!C \,||\, l$ to W whenever x is relevant for l. Recall that an object x is said to be relevant for label l if there is a label l' occurring in W such that

1. W contains a world constraint of the form $x : C \,||\, l'$, $xRy \,||\, l'$, or $yRx \,||\, l'$.

2. l is accessible from l'.

Now, if we want to deal with constant domains, a promising approach seems to be a modification of the $\rightarrow_=$ rule according to the following idea. Suppose W to be a derived system and l, l' to be labels in W. Furthermore, let $K = (\mathcal{W}, \Gamma, K_I)$ be a Kripke structure that satisfies W. Because of the constant domain assumption we know that $x^{K_I} \in \Delta^{K_I}(w)$ for each world w in $\mathcal{W}$, whenever there is a world constraint of the form $x\!:\!D \,||\, l$, $xRy \,||\, l$, or $yRx \,||\, l$ in W. In this case we say that x is a *top-level object* in W (to distinguish it from objects occurring only inside of modal operators). If x is a top-level object in W, and if the world constraint $C = \top \,||\, l'$ occurs in W, then the $\rightarrow_=$ rule must add $x\!:\!C \,||\, l'$ to W—independently from the fact whether or not x is relevant for l' (where "relevant" is defined as in the increasing domain approach). This consideration leads us to a modified rule $\rightarrow_{='}$ to handle world constraints of the form $C = \top \,||\, l$, which is given by

$W \rightarrow_{='} \{x\!:\!C \,||\, l\} \cup W$
if x is a top-level object in W, $C = \top \,||\, l$ is in W, and $x\!:\!C \,||\, l$ is not in W.

This apparently "slight" modification of the $\rightarrow_=$ rule, however, may cause infinite chains of propagation rule applications. As an example, consider the world constraint system W that consists of the two labeled $\mathcal{ALC}_\mathcal{M}$-formulas $x_0\!:\!\top \,||\, l_0$ and $(\langle o \rangle\, \exists\, R.C) = \top \,||\, l_0$, where o is an arbitrary modality. An application of the $\rightarrow_{='}$ rule yields the derived system

$$W_1 = W \cup \{x_0\!:\!\langle o \rangle\, \exists\, R.C \,||\, l_0\},$$

and, by one application of the $\rightarrow_\diamond$ and of the $\rightarrow_\exists$ rule each, we obtain

$$W_2 = W_1 \cup \{l_0 \bowtie_o l_1, x_0\!:\!\exists\, R.C \,||\, l_1, x_0Rx_1 \,||\, l_1, x_1\!:\!C \,||\, l_1\}$$

where x_1 is a new object and l_1 is a new label. Because of the newly introduced object x_1 and the world constraint $(\langle o \rangle\, \exists\, R.C) = \top \,||\, l_0$ in W_2, the $\rightarrow_{='}$ rule is again applicable, and yields

$$W_3 = W_2 \cup \{x_1\!:\!\langle o \rangle\, \exists\, R.C \,||\, l_0\}.$$

However, to $x_1 : \langle o \rangle\, \exists\, R.C \,||\, l_0$ the same propagation rules are applicable as to $x_0 : \langle o \rangle\, \exists\, R.C \,||\, l_0$ before. This means, another new label and a new object are introduced, and so on. Note that none of the newly generated objects is ever blocked since they all have different world labels. In order to avoid such infinite chains of propagation rule applications, the definition of blocked objects must be modified such that assertions with other labels are taken into account as well.

To sum up, we have seen that the problem of how to avoid infinite chains of propagation rule applications is more complicated if we are dealing with constant domains. In particular, the above example shows that, for testing whether or not an object is blocked w.r.t. some label l, it is not sufficient to consider only $\mathcal{ALC}_{\mathcal{M}}$-formulas that are labeled with l. A straightforward generalization of the notion of blocked objects, which takes different labels into account, could be defined as follows. An object x is *constant domain blocked* (for short *cd-blocked*) by an object y w.r.t. label l in a world constraint system W iff for some label l' in W it holds that $Con_W(x,l) \subseteq Con_W(y,l')$ and $y < x$. This approach is sufficient to handle the above example correctly. However, if we want to decide whether or not the $\rightarrow_\exists$ rule must be applied to a labeled $\mathcal{ALC}_{\mathcal{M}}$-formula several problems arise, which will be illustrated by the following three examples. The first example shows that in general one must take into account $\mathcal{ALC}_{\mathcal{M}}$-formulas with more than two different labels when testing whether or not an object should be blocked.

Example 6.1 Consider the system W_1 that consists of the world constraints

$$\begin{array}{ccc} x : \exists R.\exists R.A \,||\, l_0 & y : \exists R.\exists R.A \,||\, l_1 & z' : \exists R.A \,||\, l_2 \\ \tilde{z} : \neg A \,||\, l_0 & yRz \,||\, l_1 & z'R\tilde{z} \,||\, l_2 \\ & z : \exists R.A \,||\, l_1 & \tilde{z} : A \,||\, l_2. \end{array}$$

Let us have a closer look at the labeled $\mathcal{ALC}_{\mathcal{M}}$-formula $x : \exists R.\exists R.A \,||\, l_0$. There is no R-successor of x in l_0, and x is cd-blocked in W_1 w.r.t. l_0. Since x is blocked by y, and y has an R-successor z, the idea is that x can "re-use" z as its R-successor. At first sight, this seems to be feasible. However, z itself is blocked by z'. Again, there is an R-successor $\tilde{z}$ of z', and we should like to "re-use" it as R-successor of z that is in A. For label l_1, this does not lead to problems. However, our intention was to use z also with label l_0. Here the re-using of $\tilde{z}$ as an R-successors of z that is in A leads to a contradiction since we already have a constraint $\tilde{z} : \neg A \,||\, l_0$.

The second example illustrates that information about role-successors in a world constraint system may be essential when testing whether or not an object should be blocked.

Example 6.2 Suppose a derived system W_2 to be given which, among others, contains the world constraints

$$\begin{array}{cc} x : \exists R.\forall Q.A \,||\, l_1 & y : \exists R.\forall Q.A \,||\, l_2 \\ zQz' \,||\, l_1 & yRz \,||\, l_2 \\ z' : \neg A \,||\, l_1 & z : \forall Q.A \,||\, l_2. \end{array}$$

In this world constraint system the object x is cd-blocked by y w.r.t. l_1. Nevertheless, we cannot "re-use" z as R-successor of x in l_1. In fact, this would mean that we implicitly add the labeled $\mathcal{ALC}_\mathcal{M}$-formulas $xRz \,||\, l_1$ and $z : \forall\, Q.A \,||\, l_1$. These additional world constraints, however, would cause a contradiction to the labeled $\mathcal{ALC}_\mathcal{M}$-formulas $zQz' \,||\, l_1$ and $z' : \neg A \,||\, l_1$. However, if one applies the $\rightarrow_\exists$ rule to $x : \exists\, R.\forall\, Q.A \,||\, l_1$, one obtains a new object, for which no contradictions arise. This shows that x should not be blocked in this situation.

The final example shows that the test whether or not the $\rightarrow_\exists$ rule must be applied in a world constraint system W depends on the information W (implicitly) contains about the accessibility relations of Kripke structures satisfying W.

Example 6.3 Suppose W_3 to contain, among others, the world constraints

$$\begin{array}{cc} x : \exists\, R.\langle o\rangle\, A \,||\, l_1 & y : \exists\, R.\langle o\rangle\, A \,||\, l_2 \\ x_1 : [o]\, \bot \,||\, l_1 & yRz \,||\, l_2 \\ & z : \langle o\rangle\, A \,||\, l_2. \end{array}$$

Obviously W_3 is not satisfiable. However, if we do not apply the $\rightarrow_\exists$ rule to $x : \exists\, R.\langle o\rangle\, A \,||\, l_1$, the $\mathcal{ALC}_\mathcal{M}$-satisfiability algorithm does not add the world constraints $xRx' \,||\, l_1$, $x' : \langle o\rangle\, A \,||\, l_1$, $l_1 \bowtie_o l_3$, $x' : A \,||\, l_3$, and $x_1 : \bot \,||\, l_3$ to W_3 (where x' is a new object and l_3 is a new label). Since, especially, $x_1 : \bot \,||\, l_3$ is not derived, the $\mathcal{ALC}_\mathcal{M}$-satisfiability algorithm (with cd-blocking instead of blocking) does not add a clash to W_3, i.e., does not detect the unsatisfiability of W_3.

Unfortunately, we did not yet succeed in finding an appropriate definition of cd-blocked objects in world constraints. We thus leave this definition as an open problem for the moment. Note that an alternative to restricting the applicability of the $\rightarrow_\exists$ rule by the definition of cd-blocked objects would be to restrict the applicability of the $\rightarrow_\diamond$ rule in an appropriate way. However, not surprisingly, with both approaches similar problems must be solved, and it is not yet clear how this can be achieved.

7 Conclusion

The framework for integrating modal operators into terminological knowledge representation languages presented in this paper should be seen as the starting point for developing more elaborate hybrid languages of this type. Extensions in at least two directions will be necessary.

First, for the adequate representation of notions like belief and time, the basic modal logic K is not sufficient. Instead, one must consider modalities that satisfy appropriate modal axioms. A well-known example is the use of KD45 for modeling the beliefs of agents. For the case where modal operators occur only in front of terminological and assertional axioms, an integration of KD45 modal operators has already been considered in [16].

Second, the multi-dimensionality of our language has not really been made us of. In fact, we have seen that with respect to satisfiability there is no difference between the ν-dimensional and the corresponding 1-dimensional case (Corollary 5.4). We have introduced a multi-dimensional framework since it is more flexible. In an extended language, different dimensions could satisfy different modal axioms (e.g., KD45 in the belief dimension, and at least S4 in the time dimension).[4] In addition, one might want to specify certain interactions between different dimensions such as independence of one dimension from certain other dimensions.

The reason for considering a simplified framework without any of these extensions in the present paper is that in this context it is possible to design a rather intuitive calculus for satisfiability. Also, the proof of soundness, termination and completeness of this calculus is still relatively short and comprehensible. For this reason, we claim that this calculus can serve as a basis for satisfiability algorithms for more complex languages.

Another topic of future research will be investigating the constant domain assumption and its algorithmic ramifications.

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