Qualitative Reasoning About Convex Relations

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Abstract Various calculi have been designed for qualitative constraint-based representation and reasoning. Especially for orientation calculi, it happens that the well-known method of algebraic closure cannot decide consistency of constraint networks, even when considering networks over base relations (= scenarios) only. We show that this is the case for all relative orientation calculi capable of distinguishing between "left of" and "right of". Indeed, for these calculi, it is not clear whether efficient (i.e. polynomial) algorithms deciding scenario-consistency exist.
As a partial solution of this problem, we present a technique to decide global consistency in qualitative calculi. It is applicable to all calculi that employ convex base relations over the real-valued space $\mathbb{R}^n$ and it can be performed in polynomial time when dealing with convex relations only. Since global consistency implies consistency, this can be an efficient aid for identifying consistent scenarios. This complements the method of algebraic closure which can identify a subset of inconsistent scenarios.
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1 Introduction

Since the work of [1] on temporal intervals, constraint calculi have been used to model a variety of aspects of space and time in a way that is both qualitative (and thus closer to natural language than quantitative representations) and computationally efficient (by appropriately restricting the vocabulary of rich mathematical theories about space and time). For example, the well-known region connection calculus by [2] allows for reasoning about regions in space. Applications include geographic information systems, human-machine interaction, and robot navigation.

Efficient qualitative spatial reasoning mainly relies on the *algebraic closure* algorithm. It is based on an algebra of (often binary) relations: using relational composition and converse, it refines (basic) constraint networks in polynomial time. If algebraic closure detects an inconsistency, the original network is surely inconsistent. If no inconsistency is detected, for some calculi, this implies consistency of the original network — not for all calculi, though.

Orientation calculi focus on relative directions in Euclidean space, like "to the left of", "to the right of", "in front of", or "behind of". They face two difficulties: often, these calculi employ *ternary* relations, for which the theory is much less developed than for binary ones. Moreover, in this work, we show that algebraic closure can badly fail to approximate the decision of consistency of constraint networks. Hence, we look for alternative ways of tackling the consistency problem. We both refine the algebraic closure method by using compositions of higher arities, and present a polynomial decision procedure for global consistency of constraint networks that consist of convex relations. These two methods approximate consistency from below *and* above.

2 Qualitative Calculi

Qualitative calculi are employed for representing knowledge about a domain using a finite set of labels, so-called base relations. Base relations partition the domain into discrete parts. One example is distinguishing points on the time line by binary relations such as "before" or "after". A qualitative representation only captures membership of domain objects in these parts. For example, it can be represented that time point A occurs before B, but not how much earlier nor at which absolute time. Thus, a qualitative representation abstracts, which is particularly helpful when dealing with infinite domains like time and space that possess an internal structure like for example $\mathbb{R}^n$.

In order to ensure that any constellation of domain objects is captured by exactly one qualitative relation, a special property is commonly required:

Definition 1. *Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a set of n-ary relations over a domain $\mathcal{D}$. These relations are said to be* jointly exhaustive and pairwise disjoint (JEPD)*, if they satisfy the properties*

1. $\forall i, j \in \{1, \ldots, k\}$ *with* $i \neq j \ : \ B_i \cap B_j = \emptyset$
2. $\mathcal{D}^n = \bigcup_{i \in \{1,\ldots,k\}} B_i$

For representing uncertain knowledge within a qualitative calculus, e.g., to represent that objects $x_1, x_2, \ldots, x_n$ are either related by relation B_i *or* by relation B_j, *general relations* are introduced.

Definition 2. *Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a set of n-ary relations over a domain $\mathcal{D}$. The set of* general relations *$\mathcal{R}_\mathcal{B}$ (or simply $\mathcal{R}$) is the powerset $\mathcal{P}(\mathcal{B})$. The semantics of a relation $R \in \mathcal{R}_\mathcal{B}$ is defined as follows:*

$$R(x_1, \ldots, x_n) :\Leftrightarrow \exists B_i \in R, B_i(x_1, \ldots, x_n)$$

In a set of base relations that is JEPD, the empty relation $\emptyset \in \mathcal{R}_\mathcal{B}$ is called the *impossible relation*. Reasoning with qualitative information takes place on the symbolical level of relations $\mathcal{R}$, so we need special operators that allow us to manipulate qualitative knowledge. These operators constitute the algebraic structure of a qualitative calculus.

2.1 Algebraic Structure of Qualitative Calculi

The most fundamental operators in a qualitative calculus are those for relating qualitative relations in accordance to their set-theoretic disjunctive semantics. So, for $R, S \in \mathcal{R}$, intersection ($\cap$) and union ($\cup$) are defined canonically. The set of general relations is closed under these operators. Set-theoretic operators are independent of the calculus at hand, further operators are defined using the calculus semantics.

Qualitative calculi need to provide operators for interrelating relations that are declared to hold for the same set of objects but differ in the order of arguments. Put differently, we need operators which allow us to change perspective. For binary calculi only one operator needs to be defined:

Definition 3. *The converse ($^\smile$) of a binary relation R is defined as:*

$$R^\smile := \{(x_2, x_1) | (x_1, x_2) \in R\}$$

Ternary calculi require more operators to realize all possible permutations of three variables. The three commonly used operators are shortcut, homing, and inverse:

Definition 4. *Permutation operators for ternary calculi:*

$$\begin{aligned} INV(R) &:= \{ \ (y, x, z) \mid (x, y, z) \in R \ \} \quad \textit{(inverse)} \\ SC(R) &:= \{ \ (x, z, y) \mid (x, y, z) \in R \ \} \quad \textit{(shortcut)} \\ HM(R) &:= \{ \ (y, z, x) \mid (x, y, z) \in R \ \} \quad \textit{(homing)} \end{aligned}$$

Additional permutation operations can be defined, but a small basis that can generate any permutation suffices, given that the permutation operations are strong (see discussion further below) [3]. A restriction to few operations particularly eases definition of higher arity calculi.

Definition 5 ([3]). *Let $R_1, R_2, \ldots, R_n \in \mathcal{R}_\mathcal{B}$ be a sequence of n general relations in an n-ary qualitative calculus over the domain $\mathcal{D}$. Then the operation*

$$\begin{aligned} \circ\,(R_1, \ldots, R_n) := \{(x_1, \ldots, x_n) \in \mathcal{D}^n \mid \exists u \in \mathcal{D}, (x_1, \ldots, x_{n-1}, u) \in R_1, \\ (x_1, \ldots, x_{n-2}, u, x_n) \in R_2, \ldots, (u, x_2 \ldots, x_n) \in R_n\} \end{aligned}$$

is called n-ary composition.

Note that for $n = 2$ one obtains the classical composition operation for binary calculi (cp. [4]) which is usually noted as infix operator. Nevertheless different kinds of binary compositions have been used for ternary calculi, too.

2.2 Strong and Weak Operations

Permutation and composition operators define relations. Per se it is unclear whether the relations obtained by application of an operation are expressible in the calculus, i.e. whether the set of general relations $\mathcal{R}_\mathcal{B}$ is closed under an operation. Indeed, for some calculi the set of relations is not closed, there even exist calculi for which no closed set of finite size can exist, e.g. the composition operation in Freksa's double cross calculus [5].

Definition 6. *Let an n-ary qualitative calculus with relations $\mathcal{R}_\mathcal{B}$ over domain $\mathcal{D}$ and an m-ary operation $\phi : \mathcal{B}^m \to \mathcal{P}(\mathcal{D}^n)$ be given. If the set of relations is closed under ϕ, i.e. for $\forall \boldsymbol{B} \in \mathcal{B}^m \ \exists R' \in \mathcal{R}_\mathcal{B} \ : \ \phi(\boldsymbol{B}) = \bigcup_{B \in R'} B$, then the operation ϕ is called* strong.

In qualitative reasoning we must restrict ourselves to a finite set of relations. Therefore, if some operation is not strong in the sense of Def. 6, an upper approximation of the true operation is used instead.

Definition 7. *Given a qualitative calculus with n-ary relations $\mathcal{R}_\mathcal{B}$ over domain $\mathcal{D}$ and an operation $\phi : \mathcal{B}^m \to \mathcal{P}(\mathcal{D}^n)$, then the operator*

$$\begin{aligned} &\phi^\star : \mathcal{B}^m \to \mathcal{R}_\mathcal{B} \\ &\phi^\star(B_1, \ldots, B_k) := \{R \in \mathcal{B} | R \cap \phi(B_1, \ldots, B_k) \neq \emptyset\} \end{aligned}$$

is called a weak *operation, namely the weak approximation of ϕ.*

Note that the weak approximation of an operation is identical to the original operation if and only if the original operation is strong. Further note that any calculus is closed under weak operations. Applying weak operations can lead to a loss of information which may be critical in certain reasoning processes. In the literature the weak composition operation is usually denoted by $\diamond$.

Definition 8. *We call an m-ary relation R over $\mathbb{R}^n$ convex, if*

$$\{y \mid R\left(x_1, \ldots, x_{m-1}, y\right), \left(x_1, \ldots, x_{m-1}, y\right) \in \mathbb{R}^n\}$$

is a convex subset of $\mathbb{R}^n$.

3 Constraint Based Qualitative Reasoning

Qualitative reasoning is concerned with solving constraint satisfaction problems (CSPs) in which constraints are expressed using relations of the calculus. Definitions from the field of CSP are carried over to qualitative reasoning (cp. [6]).

Definition 9. *Let $\mathcal{R}$ be the general relations of a qualitative calculus over the domain $\mathcal{D}$. A* qualitative constraint *is a formula $R(X_1, \ldots, X_n)$ (also written $X_1 \ldots X_{n-1} \, R \, X_n$) with variables X_i taking values from the domain and $R \in \mathcal{R}$. A* constraint network *is a set of constraints. A constraint network is said to be a* scenario *if it gives base relations for all relations $R(X_1, \ldots, X_n)$ and the base relations obtained for different permutations of variables $X_1, \ldots, X_n$ must be agreeable wrt. the permutation operations.*

One key problem is to decide whether a given CSP has a solution or not. This can be a very hard problem. Infinity of the domain underlying qualitative CSPs inhibits searching for an agreeable valuation of the variables. This is why decision procedures that purely operate on the symbolic, discrete level of relations (rather than on the level of underlying domain) receive particular interest.

Definition 10. *A constraint network is called* consistent *if a valuation of all variables exists, such that all constraints are fulfilled. A constraint network is called* n-consistent ($n \in \mathbb{N}$) *if every solution for $n-1$ variables can be extended to a n variable solution involving any further variable. A constraint network is called* strongly *n-consistent, if it is m-consistent for all $m \leq n$. A CSP in n-variables is* globally consistent*, if it is strongly n-consistent.*

A fundamental technique for deciding consistency in a classical CSP is to enforce k-consistency by restricting the domain of variables in the CSP to mutually agreeable values. Backtracking search can then identify a consistent variable assignment. If the domain of some variable gets restricted to down to zero size while enforcing k-consistency, the CSP is not consistent. This procedure except for backtracking search (which is not applicable in infinite domains) is also applied to qualitative CSPs [4]. For a JEPD calculus with n-ary relations any qualitative CSP is strongly n-consistent unless it contains a constraint with the empty relation. So the first step in checking consistency would be to test $n+1$-consistency. In the case of a calculus with binary relations this would mean analyzing 3-consistency, also called *path-consistency.* This is the aim of the algebraic closure algorithm which exploits that composition lists all 3-consistent scenarios.

Definition 11. *A CSP over binary relations is called* algebraically closed *if for all variables X_1, X_2, X_3 and all relations R_1, R_2, R_3 the constraint relations*

$$R_1(X_1, X_2), \quad R_2(X_2, X_3), \quad R_3(X_1, X_3)$$

imply

$$R_3 \subseteq R_1 \diamond R_2$$

To enforce algebraic closure, the operation $R_3 := R_3 \cap R_1 \diamond R_2$ (as well as a similar operation for converses) is applied for all variables until a fixpoint is reached.

Enforcing algebraic closure preserves consistency, i.e., if the empty relation is obtained during refinement, then the qualitative CSP is inconsistent. However, algebraic closure does not mandatorily decide consistency: a CSP may be algebraically closed but inconsistent — even if composition is strong [7].

Algebraic closure has also been adapted to ternary calculi using binary composition [8]. Binary composition of ternary relations involves 4 variables, it may not be able to represent all 4-consistent scenarios though. Scenarios with 4 variables are specified by 4 ternary relations. However, binary composition $R_1 \diamond R_2 = R_3$ only involves 3 ternary relations. Therefore, using n-ary composition in reasoning with n-ary relations is more natural (cp. [3]).

4 Reasoning About Relative Orientation

In this section we give an account on findings for deciding consistency of qualitative CSPs. Our study is based on the $\mathcal{LR}$-calculus (ref. to [9]), a coarse relative orientation calculus. It defines nine base relations which are depicted in Fig. 1. The $\mathcal{LR}$-calculus deals with the relative position of a point C with respect to the oriented line from point A to point B, if $A \neq B$. The point C can be to the left of (l), to the right of (r) the line, or it can be on a line collinear to the given one and in front of (f) B, between A and B with the relation (i) or behind (b) A, further it can be on the start-point A (s) or an the end-point B (e). If $A = B$, then we can distinguish between the relations Tri, expressing that $A = C$ and Dou, meaning $A \neq C$. Freksa's double cross calculus $\mathcal{DCC}$ is a refinement of the $\mathcal{LR}$-calculus and, henceforth, our findings for the $\mathcal{LR}$-calculus can be directly applied to the $\mathcal{DCC}$-calculus as well. We give negative results on the applicability of existing approaches for qualitative reasoning and discuss how computations on the algebraic level can nevertheless be helpful. We begin with a lower bound of the complexity.

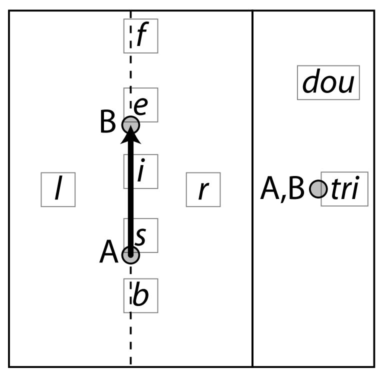

Figure 1. The nine base relations of the $\mathcal{LR}$-calculus; tri designates the case of $A = B = C$, whereas dou stands for $A = B \neq C$.

Theorem 12. *Deciding consistency of CSPs in $\mathcal{LR}$ is $\mathcal{NP}$-hard.*

Proof (sketch). In a straightforward adaption of the proof given in [10] for the $\mathcal{DCC}$ calculus, the $\mathcal{NP}$-hard problem NOT-ALL-EQUAL-3SAT can be reduced to equality of points. □

Algebraic closure usually is regarded the central tool for deciding consistency of qualitative CSPs. For the first qualitative calculi investigated (point calculus [11], Allen's interval algebra [1]) it turned out that algebraic closure decides consistency for the set of base relations, i.e. algebraic closure gives us a polynomial time decision procedure for consistency of qualitative CSPs when dealing with scenarios. This leads to the exponential time algorithm for deciding consistency of general CSPs using backtracking search to refine relations in the CSP to base relations [1]. Renz pioneered research on identifying larger sets for which algebraic closure decides consistency, thereby obtaining a practical decision procedure [12]. If however algebraic closure is too weak for deciding consistency of scenarios, no approaches are known for dealing with qualitative CSPs on the algebraic level. Unfortunately this is the case for the $\mathcal{LR}$-calculus.

Proposition 13. *All scenarios only containing the relations l and r are algebraically closed wrt. the $\mathcal{LR}$-calculus with binary composition.*

Proof. We have a look at the permutations of $\mathcal{LR}$ and see that

operation	operand	result
INV	l	r
	r	l
SC	l	r
	r	l
HM	l	l
	r	r

the set of $\{l, r\}$ is closed under all permutations. A look at the binary composition table of $\mathcal{LR}$ reveals that all compositions containing only l and r on their left hand side, always have the set $\{l, r\}$ included in their right hand side:

operand 1	operand 2	result
l	l	$\{b, s, i, l, r\}$
l	r	$\{f, l, r\}$
r	l	$\{f, l, r\}$
r	r	$\{b, s, i, l, r\}$

But with this we can conclude, that

$$R_{i,k} \diamond R_{k,j} \cap R_{i,j} \neq \emptyset$$

for all i, k, j, with $R_{n,m} \in \{l, r\}$. □

Of course not all $\mathcal{LR}$-scenarios over the variables l and r are consistent. We will show that

$$\begin{aligned}\texttt{SCEN} := \{&(A\,B\ r\ C), (A\,E\ r\ D), (D\,B\ r\ A),\\ &(D\,C\ r\ A), (D\,C\ r\ B), (D\,E\ r\ B),\\ &(D\,E\ l\ C), (E\,B\ r\ A), (E\,C\ r\ A),\\ &(E\,C\ r\ B)\}\end{aligned}$$

is algebraically closed but inconsistent. Algebraic closure directly follows from Thm. 13. We will show that any projection of this scenario to the natural domain $\mathbb{R}^2$ of the $\mathcal{LR}$-calculus yields a contradiction. Therefore we construct equations for the relations of the $\mathcal{LR}$-calculus. In $\mathbb{R}^2$ the sign of the scalar product sign($\langle \boldsymbol{X}, \boldsymbol{Y} \rangle$) determines the relative direction of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Given three points α, β and γ that are connected by an $\mathcal{LR}$-relation, we can construct a local coordinate system with origin α. It has one base vector going from α to β; we call this vector $\boldsymbol{\alpha}$. The vector orthogonal to this one and and facing to the right is called $\boldsymbol{\alpha'}$, as shown in Fig. 2. The vector from α to γ is called $\boldsymbol{\sigma}$. With this we

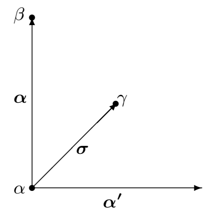

Figure 2. Constructing equations

get that $(\alpha\,\beta\;r\;\gamma)$ is true iff $\langle \boldsymbol{\alpha'}, \boldsymbol{\sigma} \rangle > 0$, and $(\alpha\,\beta\;l\;\gamma)$ is true iff $\langle \boldsymbol{\alpha'}, \boldsymbol{\sigma} \rangle < 0$, and of course we know that the points α, β, and γ are different points in these cases. The vectors $\boldsymbol{\alpha'}$ and $\boldsymbol{\sigma}$ are described by

$$\boldsymbol{\alpha'} = \begin{pmatrix} y_\beta - y_\alpha \\ x_\alpha - x_\beta \end{pmatrix}, \boldsymbol{\sigma} = \begin{pmatrix} x_\gamma - x_\alpha \\ y_\gamma - y_\alpha \end{pmatrix}.$$

With this we get

$$(\alpha\,\beta\;r\;\gamma) \Leftrightarrow (y_\beta - y_\alpha) \cdot (x_\gamma - x_\alpha) + (x_\alpha - x_\beta) \cdot (y_\gamma - y_\alpha) > 0$$

$$(\alpha\,\beta\;l\;\gamma) \Leftrightarrow (y_\beta - y_\alpha) \cdot (x_\gamma - x_\alpha) + (x_\alpha - x_\beta) \cdot (y_\gamma - y_\alpha) < 0.$$

Scenarios of the $\mathcal{LR}$-calculus are invariant wrt. the operations of translation, rotation and scaling, this means that we can fix two points to arbitrary values, we chose to set D to $(0,0)$ and B to $(0,1)$. With this we obtain the inequations

$$x_A \cdot y_E < y_A \cdot x_E \quad (1) \qquad x_C < 0 \quad (4)$$
$$x_C \cdot y_A < y_C \cdot x_A \quad (2) \qquad x_E < 0 \quad (5)$$
$$y_E \cdot x_C < x_E \cdot y_C \quad (3) \qquad 0 < x_A \quad (6)$$

In fact more inequations are derivable, but already these ones are not jointly satisfiable and we conclude:

Theorem 14. *Classical algebraic closure does not enforce scenario consistency for the $\mathcal{LR}$-calculus.*

Proof. We consider the algebraically closed $\mathcal{LR}$ scenario `SCEN` and the inequations (1) to (6) that we derived when projecting it into $\mathbb{R}^2$, the intended domain of $\mathcal{LR}$. From inequations (1), (6), (4), (5) and (3) we obtain

$$\frac{x_E \cdot y_C}{x_C} < y_E < \frac{y_A \cdot x_E}{x_A}$$

and again using inequations (6), (4) and (5) we get

$$y_C \cdot x_A < x_C \cdot y_A$$

contradicting (2). Hence our scenario is not consistent. □

As discussed earlier ternary composition is more natural for ternary calculi than binary composition. Therefore we examined the ternary composition table of the $\mathcal{LR}$-calculus[3] and conclude:

Theorem 15. *Algebraic closure wrt. ternary composition does not enforce scenario consistency for the $\mathcal{LR}$-calculus.*

Proof. Let us have a closer look at the ternary composition operation wrt. the relations contained in `SCEN`, namely the relation l and r. Recall that the set $\{l, r\}$ of $\mathcal{LR}$-relations is closed under all permutation operations. So we only need to consider the fragment of the composition table with triples over l, r:

$$\begin{aligned} &\diamond(r,r,r) = \{r\}, && \diamond(r,r,l) = \{b,r,l\},\\ &\diamond(r,l,r) = \{f,r,l\}, && \diamond(r,l,l) = \{i,r,l\},\\ &\diamond(l,r,r) = \{i,r,l\}, && \diamond(l,r,l) = \{f,r,l\},\\ &\diamond(l,l,r) = \{b,r,l\}, && \diamond(l,l,l) = \{l\}. \end{aligned}$$

We see that any composition that contains r as well as l in the triple on the left-hand side yields a superset of $\{r, l\}$ on the right-hand side. So all composable triples that have both l and r on their left hand side cannot yield an empty set while applying algebraic closure. So, we have to investigate how the compositions $\diamond(l,l,l)$ and $\diamond(r,r,r)$ are used when enforcing algebraic closure. Enumerating all composable triples $(X_1\, X_2\, r_1\, X_4), (X_1\, X_4\, r_2\, X_3), (X_4\, X_2\, r_3\, X_3)$ and their respective refinement relation $(X_1\, X_2\, r_f\, X_3)$ yields a list with 18 entries shown in Appendix A. All of those entries list l as refinement relation whenever composing $\diamond(l,l,l)$ and analogously for r. Thus, no refinement is possible, and the given scenario is algebraically closed wrt. ternary composition. □

We believe that advancing to even higher arity composition will not provide us with a sound algebraic closure algorithm. It turns out, however, that moving to a certain level of k-consistency does indeed make a change.

[3] Such a table is available via the qualitative reasoner `SparQ`. (ref. to http://www.sfbtr8.spatial-cognition.de/project/r3/sparq/)

Remark 16. Of course it is theoretically possible to solve these systems of inequations by quantifier elimination, or by the more optimized Cylindrical Algebraic Decomposition (CAD). Unfortunately the CAD algorithm has a double exponential worst case running time (even though this can be reduced to polynomial running time with a optimal choice of the involved projections). Our experiments with CAD tools unfortunately were quite disillusioning, since those tools choked on our problems mainly because of the large number of involved variables (consider that each point in our scenarios introduces 2 variables in our systems of inequalities).

5 Deciding Global Consistency

In this section we will generalize a technique from [13] and we will show that this generalization decides global consistency for arbitrary CSPs over m-ary convex relations over a domain $\mathbb{R}^n$. The resulting theorem transfers Thm. 5 of [14] from classical constraint satisfaction to qualitative spatio-temporal reasoning.

Theorem 17 (Helly [15]). *Let S be a set of convex regions of the n-dimensional space $\mathbb{R}^n$. If every $n+1$ elements in S have a nonempty intersection then the intersection of all elements of S is nonempty.*

Theorem 18. *A CSP over m-ary convex relations over a domain $\mathbb{R}^n$ is globally consistent, i.e. k-consistent for all $k \in \mathbb{N}$, if and only if it is strongly $((m-1) \cdot (n+1)+1)$-consistent.*

Proof. In the first step of this proof consider an arbitrary CSP over convex m-ary relations that is strongly $(m-1) \cdot (n+1)+1$ consistent. By induction on k, which is the number of variables that can be instantiated in a strongly consistent way, we show that it is $k+1$ consistent for an arbitrary k. Assume that for each tuple $(X_1, \ldots, X_k)$ of these variables a consistent valuation $(z_1, \ldots, z_k)$ exists. For this purpose we define sets

$$p_s\left(\left(z_{i_1}, \ldots, z_{i_{m-1}}\right), R_{i_1,\ldots,i_s,k+1,i_{s+1},\ldots,i_{m-1}}\right) = \{z \mid R_{i_1,\ldots,i_s,k+1,i_{s+1},\ldots,i_{m-1}}\left(z_{i_1}, \ldots, z_{i_s}, z, z_{i_{s+1}}, \ldots, z_{i_{m-1}}\right)\}$$

with $1 \leq i_j \leq k$ and $1 \leq s \leq m-1$. By assumption, these are sets of convex regions of the particular space defined by the assignment of the variables $(X_1, \ldots, X_k) \mapsto (z_1, \ldots, z_k)$ and the particular relation $R_{i_1,\ldots,i_s,k+1,i_{s+1},\ldots,i_{m-1}}$. Let

$$\mathbf{P} = \{p_s\left(\left(z_{i_1}, \ldots, z_{i_{m-1}}\right), R_{i_1,\ldots,i_s,k+1,i_{s+1},\ldots,i_{m-1}}\right) \mid 1 \leq s \leq m-1 \land 1 \leq i_j \leq k\}$$

be the set of all such convex regions. Observe that $n+1$ tuples of elements of $\mathbf{P}$ are induced by constraints containing up to $(m-1) \cdot (n+1)$ different variables.

By strong $((m-1)\cdot(n+1)+1)$-consistency we know that the intersection of all these regions is non-empty. The application of Helly's Theorem yields

$$\bigcap_{p\in\mathbf{P}} p \neq \emptyset.$$

Hence a valuation for $k+1$ variables exists. The second step of this proof is trivial, since global consistency implies k-consistency for all $k \in \mathbb{N}$. □

In [7, Prop. 1] it was shown that whether composition is weak or strong is independent of the property of algebraic closure to decide consistency. However, in some cases, these two properties *are* related:

Theorem 19. *In a binary calculus over the real line that*

1. *has only* 2*-consistent relations*
2. *and has strong binary composition*

algebraic closure decides consistency of CSPs over convex base relations.

Proof (Proof sketch). By Thm. 18 we know that strong 3-consistency decides global consistency. Since composition is strong, algebraic closure decides 3-consistency and, since we have 2 consistency, it decides strong 3-consistency too. Thus algebraically closed scenarios are either inconsistent (containing the empty relation) or globally consistent. Put differently, global consistency and consistency coincide. □

Corollary 20. *For CSPs over convex* $\{\mathcal{LR}, \mathcal{DCC}\}$*-relations strong* 7*-consistency decides global consistency.*

Proof. Follows directly from Thm. 18 for both calculi. □

Corollary 21. *Global consistency of scenarios in convex* $\{\mathcal{LR}, \mathcal{DCC}\}$*-relations is polynomially decidable.*

Proof. Compute the set of strongly 7-consistent scenarios in constant time (e.g. using quantifier elimination[4]). The given scenario is strongly 7-consistent iff all 7-point subscenarios are contained in the set of strongly 7-consistent scenarios. By Thm. 18 this decides global consistency. □

Unfortunately consistency and global consistency are not equivalent in the $\mathcal{LR}$-calculus.

Proposition 22. *For the* $\mathcal{LR}$*-calculus not every consistent scenario is globally consistent.*

[4] Here we just want to state the computation is possible, we do not claim to suggest a practical method though.

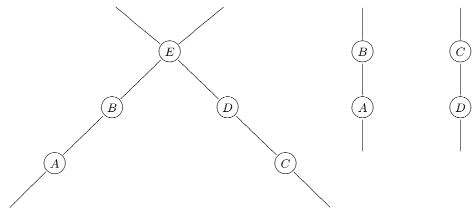

Figure 3. Illustration for Prop. 22

Proof. Consider the consistent scenario

$$\{(A\,B\ r\ C), (A\,B\ r\ D), (C\,D\ l\ A)$$
$$(C\,D\ l\ B), (A\,B\ f\ E), (C\,D\ f\ E)\}$$

which has a realization as shown in Fig. 3 (left), the lines $\overline{AB}$ and $\overline{CD}$ intersect. Now consider the sub-CSP in the variables A, B, C, and D with the solution shown in Fig. 3 (right). We see that the lines $\overline{AB}$ and $\overline{CD}$ are parallel, but the constraints $(A\,B\ f\ E)$ and $(C\,D\ f\ E)$ demand that the point E is on the line $\overline{AB}$ as well as on the line $\overline{CD}$, hence the given scenario is not 5-consistent, and so it is not globally consistent. □

6 Discussion & Conclusion

We have shown that for relative orientation calculi capable of distinguishing between "left of" and "right of" like the $\mathcal{LR}$-calculus, the composition table alone is not sufficient for deciding consistency of qualitative scenarios. We have argued that binary composition in ternary calculi in general does not provide sufficient means for generalizing algebraic closure to ternary calculi. Instead ternary composition is required. However, advancing to ternary composition which can list 4-consistent scenarios and thus allows us to generalize algebraic closure is still not sufficient for deciding consistency. This is a remarkable result that has implications to several relative orientation calculi to which the given proofs can be transferred:

- $\mathcal{LR}$ calculus [16]
- Dipole calculus [17]
- $\mathcal{OPRA}$ calculus family [18]
- Double-cross calculus ($\mathcal{DCC}$) [19]

To conclude, at the time being we have no practical method for deciding consistency in any of the listed relative orientation calculi. This may lead to a dramatic impact on qualitative spatial reasoning: The highly structured spatial domain does not yet help us to implement more effective reasoning algorithms than for general logical reasoning. So far the only backbone for reasoning with relative information is given by a logic-based approach [20].

In future work the practical utility of the presented polynomial-time decision procedure given by Cor. 21 for global consistency needs to be analyzed. While the general problem of deciding consistency of constraint satisfaction problems in $\mathcal{LR}$ is $\mathcal{NP}$-hard, it is likely to be easier for scenarios. Therefore, our future work will be involved with singling out tractable problem classes and we aim at developing a method for deciding consistency of qualitative constraint satisfaction problems contained in $\mathcal{NP}$, possibly finding a polynomial-time method for scenarios.

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A Table of composable l/r triples

$$\begin{array}{ccccc} (A\,C\,l\,B) & (A\,B\,l\,D) & (B\,C\,l\,D) & & (A\,C\,l\,D) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (A\,C\,l\,E) & (A\,E\,l\,D) & (E\,C\,l\,D) & & (A\,C\,l\,D) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (A\,C\,l\,B) & (A\,B\,l\,E) & (B\,C\,l\,E) & & (A\,C\,l\,E) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (E\,A\,l\,B) & (E\,B\,l\,C) & (B\,A\,l\,C) & & (E\,A\,l\,C) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (C\,D\,l\,B) & (C\,B\,l\,A) & (B\,D\,l\,A) & & (C\,D\,l\,A) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (C\,D\,l\,E) & (C\,E\,l\,A) & (E\,D\,l\,A) & & (C\,D\,l\,A) \end{array}$$

$$\begin{array}{cccccc}
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& l, & l, & l) & \cap & \{l\} = \{l\} \\
\hline
 & (C\,E\,l\,B) & (C\,B\,l\,A) & (B\,D\,l\,A) & & (C\,E\,l\,A) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& l, & l, & l) & \cap & \{l\} = \{l\} \\
\hline
 & (E\,C\,r\,B) & (E\,B\,r\,A) & (B\,C\,r\,A) & & (E\,C\,r\,A) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (D\,A\,l\,B) & (D\,B\,l\,C) & (B\,A\,l\,C) & & (D\,A\,l\,C) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& l, & l, & l) & \cap & \{l\} = \{l\} \\
\hline
 & (D\,A\,l\,E) & (D\,E\,l\,C) & (E\,A\,l\,C) & & (D\,A\,l\,C) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& l, & l, & l) & \cap & \{l\} = \{l\} \\
\hline
 & (A\,D\,r\,B) & (A\,B\,r\,C) & (B\,D\,r\,C) & & (A\,D\,r\,C) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (A\,D\,r\,E) & (A\,E\,r\,C) & (E\,D\,r\,C) & & (A\,D\,r\,C) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (A\,E\,r\,B) & (A\,B\,r\,C) & (B\,E\,r\,C) & & (A\,E\,r\,C) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (C\,A\,r\,B) & (C\,B\,r\,E) & (B\,A\,r\,E) & & (C\,A\,r\,E) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (C\,A\,r\,E) & (C\,E\,r\,D) & (E\,A\,r\,D) & & (C\,A\,r\,D) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (C\,A\,r\,B) & (C\,B\,r\,D) & (B\,A\,r\,D) & & (C\,A\,r\,D) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (D\,C\,r\,B) & (D\,B\,r\,A) & (B\,C\,r\,A) & & (D\,C\,r\,A) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
 & (D\,C\,r\,E) & (D\,E\,r\,A) & (E\,C\,r\,A) & & (D\,C\,r\,A) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\}
\end{array}$$